AN EXTENDED DUAL PARAMETERIZATION METHOD TO CONSTRAINED LEAST SQUARE LAGUERRE FILTER DESIGN

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ABSTRACT

The design of peak-constrained weighted least-square error (PCWLSE) Laguerre filters is of interest in signal processing applications. The formulation of such Laguerre filters ends up as a semi-infinite problem with two uncountable variables. Here, an efficient method is proposed to deal with this. Therefore, one can design PCWLSE Laguerre filters to satisfy all the continuous constraints of the problem. Furthermore, the desired phase response is also guaranteed. The proposed method has been developed based on exploiting the dual parameterization scheme derived directly from the Karush-Kuhn-Tucker condition. This approach may also be used to design similar asymmetric FIR filters.

1. INTRODUCTION

The design of ideal filters by FIR filter structures usually results in filters that need a large number of taps. However, Laguerre filters introduce a general form of filter structure that can be substituted for FIR filters with a significant reduction in the number of taps required to achieve the same performance. They are especially well suited in applications where narrow-band filters and consequently a large number of FIR filter taps are needed [1]. Moreover, reducing the number of taps is not only beneficial in VLSI implementations, it is also favorable in filter design procedures, because computational schemes for filter design become extremely memory intensive as the number of taps increases.

Digital Laguerre filters have been conventionally designed according to two different criteria: least square error (L_2 norm) and Chebyshev error (L_∞ norm). However, it turns out that filters that are designed according to a trade-off between L_2 and L_∞ norms have more desirable properties than ones designed by either criterion alone. They are also more suitable for a vast number of signal processing applications [2]. These filters are called peak-constraints weighted least-square (PCWLSE) filters. PCWLSE filter design can be formulated as a semi-infinite quadratic programming optimization problem [3]-[5]. In [3], an extended active set method has been proposed to solve the problem of FIR window design. This method has been used in [6] to optimize antenna arrays, and used in [4] to solve the problem of PCWLSE design of digital Laguerre filters. Also the magnitude restriction of the complex Chebyshev error as the constraint in [3] and [4] produces an additional uncountable variable. Therefore, there are two uncountable variables which convert the problem to a semi-infinite dimensional case. To solve this problem, one of the variables (frequency) is considered to be discretized and then the problem is solved using an extended active set method on a dense grid of frequencies. In [5], the problem of symmetric FIR filters is investigated. Thus, the only uncountable variable of the problem that changes the problem to a semi-infinite case is the frequency. Then the problem has been solved using an efficient algorithm. This algorithm guarantees achieving an optimum solution (global minimum) if the problem satisfies the continuous constraints at points between the dense grid of frequencies.

Here, we present a new and efficient method to design the Laguerre filter counterpart of the above problem. We will show that for the first time, one can design a peak constrained-weighted least-square Laguerre filter such that its optimum solution satisfies the continuous constraints of both frequency and the other uncountable variable of the problem. However, there are some significant differences between this work and [5]. First, the problem has been formulated to deal with Laguerre filters rather than FIR filters. Second, the procedure used here to extend the dual problem of finite dimensional space to the semi-infinite dimensional case is completely different from the one used in [5]. Third, the general procedure proposed here also differs from the one proposed in [5], which makes the new proposed algorithm more efficient.

It is notable that the new proposed approach is also able to handle the design of constrained asymmetric FIR filters. This is reasonable because the Laguerre filter structures, which are generally nonsymmetrical, can be converted to conventional asymmetric FIR filters by setting the Laguerre parameter equal to zero [4]. More on this will be presented as a separate work in the future.

2. PROBLEM STATEMENT

The frequency response of the Laguerre filter of order N with real coefficients $\xi \in \Re^N$ is characterized by [4]:

$$H(e^{j\omega}) = L^T(e^{j\omega})\xi \tag{1}$$

where $L(e^{j\omega})$ is an $N \times 1$ vector with elements:

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$$L_l(e^{j\omega}) = \sqrt{(1-b^2)}\,\frac{(e^{-j\omega}-b)^l}{(1-be^{-j\omega})^{l+1}} \quad ,l = 0\,,1\,,...,\ N-1 \tag{2}$$

wherein b denotes the Laguerre parameter and $|b| < 1$ guarantees the stability of the filter. Let $H_d(e^{j\omega})$ denote the desired frequency response and Ω_P and Ω_S be the passband and stopband, respectively, which are both compact and uncountable subsets of $[0,\pi]$. The PCWLSE design problem can be stated as follows [4]:

$$\min_{\xi\in\Re^N} \|E\|_2 \quad \text{subject to} \tag{3}$$
$$\left|H(e^{j\omega}) - H_d(e^{j\omega})\right| < \delta(\omega) \quad ,\omega \in \Omega_{PS} = \Omega_P \cup \Omega_S$$

where

$$\|E\|_2 = w_P \int_{\Omega_P} \left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega + w_S \int_{\Omega_S} \left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega \tag{4}$$

In (4) w_p and w_s are weighting coefficients for the passband and stopband, respectively, with $w_S / w_p >> 1$ [2]. Also $\delta(\omega)$ denotes the maximum allowable error at each frequency point. Using the real rotation theorem [7], problem (3) is converted to the following semi-infinite quadratic programming problem:

$$\min_{\xi\in\Re^N} \frac{1}{2}\xi^T \psi_r \xi + \varphi^T \xi \quad \text{subject to} \tag{5}$$
$$\alpha^T(\omega,\lambda)\xi \le \gamma(\omega,\lambda) \quad ,\omega \in \Omega_{PS}\,,\lambda \in [0,2\pi]$$

where

$$\psi_r = \operatorname{Re}\{\psi\} \tag{6}$$
$$\alpha(\omega,\lambda) = \operatorname{Re}\left\{L(e^{j\omega})e^{j\lambda}\right\} \tag{7}$$
$$\gamma(\omega,\lambda) = \delta(\omega) + \operatorname{Re}\left\{H_d(e^{j\omega})e^{j\lambda}\right\} \tag{8}$$

and $\psi \in \Re^{N\times N}$ is a Toeplitz, Hermitian, positive definite matrix, whose elements $\psi(m,n)$ are defined as:

$$\psi(m,n) = \left[2w_p \int_{\Omega_P} \frac{e^{j\omega}(1-be^{j\omega})^{m-n-1}}{(e^{j\omega}-b)^{m-n+1}} d\omega + 2w_S \int_{\Omega_S} \frac{e^{j\omega}(1-be^{j\omega})^{m-n-1}}{(e^{j\omega}-b)^{m-n+1}} d\omega\right](1-b^2) \tag{9}$$

and $\varphi \in \Re^N$ is a vector with elements:

$$\varphi(k) = -2\operatorname{Re}\left\{w_P \int_{\Omega_P} H_d(e^{j\omega}) L_k^*(e^{j\omega}) d\omega + w_S \int_{\Omega_S} H_d(e^{j\omega}) L_k^*(e^{j\omega}) d\omega\right\} \tag{10}$$

in which $\operatorname{Re}\{.\}$ and $*$ denote the real part and complex conjugate, respectively. The integrals in (9) and (10) can be evaluated using numerical methods such as the Newton-Cotes formula or Simpson's rule.

3. PROBLEM SOLUTION

3.1. Slater's Constraint Qualification

The optimum solution to problem (5) is found by the extended dual parameterization method if we assume the following condition is satisfied, which is called Slater's constraint qualification.

Assumption1: Let $\Psi = \Omega_{PS} \times [0,2\pi]$ and $g(\xi,\omega,\lambda) = \alpha^T(\omega,\lambda)\xi - \gamma(\omega,\lambda) \quad \forall(\omega,\lambda) \in \Psi$. Let $\bar{\xi} \in \Re^N$ be an arbitrary point. Each $g \in \{g(\bar{\xi},\omega,\lambda) = 0,\ (\omega,\lambda) \in \Psi\}$ is pseudoconvex at $\bar{\xi}$, each $g \in \{g(\bar{\xi},\omega,\lambda) \neq 0,\ (\omega,\lambda) \in \Psi\}$ is continuous at $\bar{\xi}$, and there exists a $\xi \in \Re^N$ such that $g \in \{g(\xi,\omega,\lambda) < 0,\ \forall(\omega,\lambda) \in \Psi\}$.

Due to the continuity and pseudoconvexity of the constraints used in the problem, the corresponding statements of Assumption 1 are held to be true. The last statement of the assumption is also true if the transition band and the maximum allowable error are not chosen too narrow or too small, respectively. Assumption 1 can be checked by solving the following semi-infinite linear optimization problem:

$$\min_{(\nu,\xi)\in\Re^+\times\Re^N} \nu \quad \text{subject to} \tag{11}$$
$$\frac{1}{\delta(\omega)}\left[\alpha^T(\omega,\lambda)\xi - \operatorname{Re}\left\{H_d(e^{j\omega})\right\}\right] \le \nu \quad ,\omega \in \Omega_{PS}, \lambda \in [0,2\pi]$$

where $\Re^+$ denotes the set of positive real numbers. If $\nu_{\min} \le 1$, Assumption 1 is satisfied. Otherwise the problem is not feasible and also the assumption is not true. The semi-infinite linear programming problem (11) can be solved by extension of the simplex algorithm [8].

3.2 Extended Dual Parameterization Method

Let ρ be a regular Borel measure and $M^+(\Psi)$ denote the space of all positive regular Borel measures on Ψ defined in Assumption 1. The necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions for problem (5) are stated as follows ([3], [4], [6]):

$$\begin{aligned} &\psi_r \xi + \varphi + \int_\Psi \alpha(\omega,\lambda) d\rho(\omega,\lambda) = 0 \\ &\int_\Psi \left[\alpha^T(\omega,\lambda)\xi - \gamma(\omega,\lambda)\right] d\rho(\omega,\lambda) = 0 \\ &\alpha^T(\omega,\lambda)\xi - \gamma(\omega,\lambda) \le 0 \\ &\rho \ge 0\ ,(\omega,\lambda) \in \Psi \end{aligned} \tag{12}$$

Let ξ be feasible. Using Caratheodory's theorem, the optimum and non-unique measure ρ satisfying (12) can be represented by a measure with finite support (counting or atomic measure) which is not greater than N points [3], [4], [6]. Hence the KKT conditions turn into:

$$\psi_r \xi + \varphi + \sum_{i=1}^{k} \rho_i \alpha(\omega_i,\lambda_i) = 0 \tag{13}$$
$$\rho_i \ge 0 \quad ,(\omega_i,\lambda_i) \in B$$

where

$$B = \left\{(\omega_r,\lambda_r) \in \Psi \middle| \alpha^T(\omega_r,\lambda_r)\xi = \gamma(\omega_r,\lambda_r)\right\} \quad , r = 1\,,2\,,...,k \tag{14}$$

Note that the values of λ_r related to each ω_r are determined by:

$$\lambda_r = -\arg\left[H(e^{j\omega_r}) - H_d(e^{j\omega_r})\right] \tag{15}$$

Now, the problem has been converted to finite dimensional space. The Lagrangian function related to this problem can be stated as follows [9, pp.167-168]:

$$\mathcal{L}(\xi,\omega_i,\lambda_i,U) = \frac{1}{2}\xi^T\psi_r\xi + (\varphi + AU)^T\xi - U^T\Gamma \tag{16}$$

where

$$U = [\rho_1\ \rho_2\ ...\ \rho_k]^T \tag{17}$$

and $A \in \Re^{N\times k}$, $\Gamma \in \Re^{N\times 1}$ in which:

$$A = \alpha(\omega_r,\lambda_r),\quad \Gamma = \gamma(\omega_r,\lambda_r),\quad (\omega_r,\lambda_r) \in B \tag{18}$$

The elements of matrix A are:

$$A(m,n) = \mathrm{Re}[L_m(e^{j\omega n})e^{j\lambda n}],\ (\omega_n,\lambda_n) \in B, \quad m = 1,2,...,N \quad ,n = 1,2,...,k \tag{19}$$

Note that vector U plays the role of Lagrange multipliers defined in finite dimensional cases.

Due to the positive definiteness of ψ_r, the objective function of problem (5) is convex. Therefore, under Assumption 1, no duality gap exists, and the optimal value of problem (5) is equal to the optimal value of the Lagrangian dual problem. This is known as strong duality theorem (see [9]). i.e.

$$\frac{1}{2}\xi_{opt}^T\psi_r\xi_{opt} + \varphi^T\xi_{opt} = \max_{\rho_i\in[0,\infty)}\ \min_{\xi\in\Re^N}\ \mathcal{L}(\xi,\omega_i,\lambda_i,U)\ ,(\omega_i,\lambda_i)\in B \tag{20}$$

where the second term in (20) can be represented as:

$$\max_{(\omega_i,\lambda_i,\rho_i)\in X}\ \min_{\xi\in\Re^N}\ \mathcal{L}(\xi,\omega_i,\lambda_i,U) \tag{21}$$

Here, ξ_{opt} is the optimal solution of problem (5) and $X = \Psi\times[0,\infty)$. The Lagrangian function is also convex with respect to ξ. Thus, the necessary and sufficient condition to obtain minimum of $\mathcal{L}(\xi,\omega_i,\lambda_i,U)$ is:

$$\nabla_\xi\ \mathcal{L}(\xi,\omega_i,\lambda_i,U) = 0 \tag{22}$$

or

$$\xi = -\mathrm{P}_{\psi_r^{-1}}(\varphi + AU) \tag{23}$$

where $\mathrm{P}_{\psi_r^{-1}}$ denotes Moore-Penrose pseudo inverse of ψ_r [10] and we suggest it since ψ_r is usually ill conditioned. Substitution of (23) in (16) results in:

$$\mathcal{L}(\omega_i,\lambda_i,U) = -\frac{1}{2}(\varphi + AU)^T\mathrm{P}_{\psi_r^{-1}}(\varphi + AU) - U^T\Gamma \tag{24}$$

Thus, the optimum solution to the "max min" problem stated in (20) is achieved from:

$$\max_{(\omega_i,\lambda_i,\rho_i)\in X} \mathcal{L}(\omega_i,\lambda_i,U) \tag{25}$$

The problem (25) can be stated as:

$$\min_{(\omega_i,\lambda_i,\rho_i)\in X} -\mathcal{L}(\omega_i,\lambda_i,U) \tag{26}$$

This minimization can be easily solved by the Matlab optimization toolbox. If the initial values of (ω_i,λ_i,U) are near to the global minimum of $-\mathcal{L}(\omega_i,\lambda_i,U)$, then Matlab is able to find the global minimum, otherwise grids with more points are needed. Here, are the basic steps of the algorithm:

Step 1: Check if Assumption 1 is satisfied. This is done by solving the system of equations in (11). If the assumption is true, go to Step 2.

Step 2: Solve the problem stated in (5) by discretizing Ω_{PS} for ω and $\lambda_i \in [0,2\pi]$. Then, find $(\omega_i,\lambda_i)\in B$. These are suboptimal solutions to problem (26) that can be considered as initial values of ω and λ. In order to find good initial values for ρ_i related to the set of suboptimal values $(\omega_i,\lambda_i)\in B$, we need to solve the following problem:

$$\min_{\rho_i\in[0,\infty)} -\mathcal{L}(U) \tag{27}$$

This minimization is only performed on the variable ρ_i and can be easily minimized (see Remark 1). After this initialization step, refinement is needed by solving the 3×k-dimensional problem (25) using the numerical gradient method in the Matlab optimization toolbox. Let $(\omega_i,\lambda_i,\rho_i)_{opt}$ be the optimal value of this problem. Then the optimum filter coefficients can be found by using $(\omega_i,\lambda_i,\rho_i)_{opt}$ in (23).

Step 3: If the designed filter satisfies the continuous constraints of the problem, it is the optimum filter. Otherwise, one should begin Step 2 by a grid with more points to find closer initial points to the global minimum of the problem (see Theorem 3.5 of [5]).

Remark1: The minimization stated in (27), can be solved analytically as follows:

Assume that the set of $(\omega_i,\lambda_i)\in B$ are determined in Step 2. By substituting these (ω_i,λ_i) in (24), it can be shown that:

$$-\mathcal{L}(U) = \frac{1}{2}U^TA^T\mathrm{P}_{\psi_r^{-1}}AU + (\varphi^T\mathrm{P}_{\psi_r^{-1}}A + \Gamma^T)U + \frac{1}{2}\varphi^T\mathrm{P}_{\psi_r^{-1}}\varphi \tag{28}$$

$\mathcal{L}(U)$ is quadratic in U. Thus, the necessary and sufficient condition for minimization it is:

$$\nabla_U\mathcal{L}(U) = 0 \tag{29}$$

This leads us to:

$$U = -\mathrm{P}_{A^T\mathrm{P}_{\psi_r^{-1}}A}(\varphi^T\mathrm{P}_{\psi_r^{-1}}A + \Gamma^T)^T \tag{30}$$

where $\mathrm{P}_{A^T\mathrm{P}_{\psi_r^{-1}}A}$ is Moore-Penrose pseudo inverse of $A^T\mathrm{P}_{\psi_r^{-1}}A$. Due to convexity of $-\mathcal{L}(U)$ and the feasible region defined in (27) for ρ_i, set zero those negative values of ρ_i found in (30).

Remark 2: We do not include the Laguerre parameter as a variable in the optimization process as in [11]. Rather, this is determined by two successive exhaustive searches. The first is done in (11) to find the feasible Laguerre parameters and the second is performed to find the best Laguerre parameter related to the least square error. The second exhaustive search is not usually necessary because the best Laguerre parameter found in the second search is usually very close to the Laguerre parameter that corresponds to the minimum ν among the set of ν_{min} obtained from the first search.

4. Numerical Example

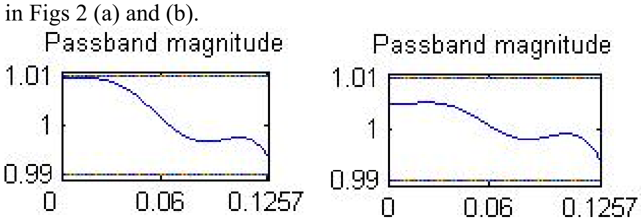
We apply the above technique to the following example.

Example: Design a Laguerre filter with the following specifications:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j18\omega}, \omega \in [0.0.04\pi] \\ 0, \omega \in [0.14\pi, \pi] \end{cases} \quad , \; w_S / w_P = 100$$

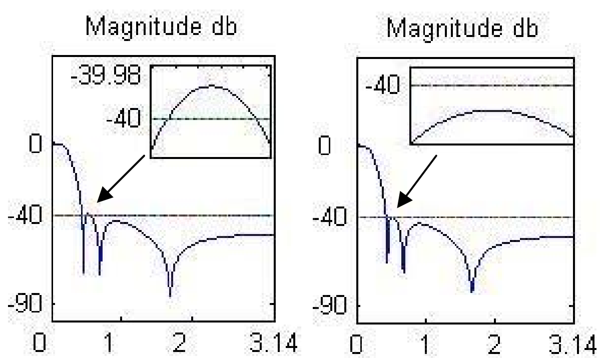
$\delta(\omega) \leq 0.01$ in both passband and stopband. Using Remark 2 the optimum Laguerre parameter with the above specifications is 0.76. Initial points were found by discretizing both $\omega \in \Omega_{PS}$ and $\lambda \in [0, 2\pi]$ into 300 points.

In this example, the Laguerre filter of order 11 that satisfies all the continuous constraints has almost the same performance as an FIR filter with 40 stages. The simulation results are shown in Figs. 1-3. Figures 1 (a) and (b) show the magnified passband magnitude responses before and after refinement, respectively. The corresponding magnitude responses in the stopband are illustrated in Figs 2 (a) and (b).



(a) (b)

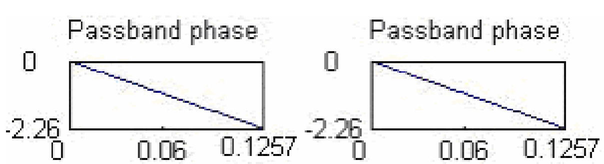
Figure 1. Passband magnitude response of (a) initial filter design and (b) after refinement.



(a) (b)

Figure 2. (a) Filter designed by initial points does not satisfy the stopband constraints at some points (total weighted least square error = 0.9738) (b) Filter designed after refinement process satisfies all the constraints of stopband (total weighted least square error = 0.9714).

Figures 3 (a) and 3 (b) are the passband phase responses before and after refinement. Both filters are almost the same. These figures show a nearly straight line with slope -18.



(a) (b)

Figure 3 Passband phase response with (a) initial points and (b) after refinement.

5. Conclusions

In this paper we devised a method to design PCWLSE Laguerre filters based on using a finite dual parameterization technique. The Laguerre filters constitute a more general form of the digital filters whose one special case is the set of FIR filters. They can be substituted for long FIR filters with almost the same performance but with significantly fewer number of taps. In design of these filters we encountered the problem of semi-infinite quadratic programming with two uncountable variables. Our proposed method was able to cope with this problem such that the resultant solution satisfies all the continuous constraints of the magnitude response. The desired phase response was also guaranteed. This approach has also led us to the design of PCWLSE asymmetric FIR filters.

6. References

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