AN EXTENDED ε-PERTURBATION METHOD TO THE PROBLEM OF SEMI-INFINITE QUADRATIC PROGRAMMING IN CONSTRAINED FIR FILTER DESIGN

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ABSTRACT

This paper is concerned with the design of peak constrained weighted least square error (PCWLSE) FIR filters. Successive use of Caratheodory's theorem and Farkas' lema leads us to a new approach which is an extended ε-perturbation method to solve the resultant semi-infinite quadratic optimization problems of this type of filter design. The approach does not need the calculation of Lagrange multipliers which may require the inversion of ill-conditioned matrixes. The new proposed method is able to deal with design of both symmetric and asymmetric FIR filters. The computational procedure is illustrated by solving an example.

1. INTRODUCTION

Digital filters are conventionally designed according to two different criteria: least square error criterion (L_2 norm) and Chebyshev error criterion (L_∞ norm). However, Adams proposed another efficient filter design criterion which is a trade-off between L_2 and L_∞ norms known as peak constrained weighted least square error (PCWLSE) and is suitable for many signal processing applications [1].

In [1-3], the problem of PCWLSE design of symmetric FIR filters with linear phase is investigated. Also some works deal with the design of asymmetric FIR filters with arbitrary phase responses [4-5]. In [1-5], the problem of PCWLSE FIR filter design has been formulated as a quadratic programming problem and has been solved by different methods. More recently, the PCWLSE design of FIR filters has been formulated as a semi-infinite quadratic programming as in [6] and [7]. These problems are called semi-infinite due to infinite number of constraints and finite number of variables. In [6] the problem has been formulated for only symmetric FIR filters and solved by minimization of the Lagrangian function. In [7] an extended active set method is applied to solve the problem of FIR window design. This approach has been used in [8] to optimize antenna arrays. Also a developed version of this method has been used in [9] to solve the problem of PCWLSE digital Laguerre filter design.

Here, we propose a new technique which is an extended version of ε-perturbation method [10] to solve the semi-infinite quadratic programming optimization problem of PCWLSE digital FIR filters. The ε-perturbation approach is based on Zoutendijk work [11] and by using the concept of near binding constraints this algorithm avoids jamming phenomenon and converges to the Karush-Kuhn-Tucker point of the problem.

Having the advantages of extended active set method described in [7-8], this new approach is very simple to implement and also does not need the calculation of Lagrange multipliers. Calculation of Lagrange multipliers usually needs inversion of some matrixes which are usually ill conditioned and the solutions are very sensitive to any small error in data. These errors may come from evaluation of problem integrations by finite summations or from approximating of any infinite and uncountable set of problems by finite and countable sets. Furthermore, compared to dual parameterization method which is an efficient algorithm used in [6] for symmetric FIR filters, the new proposed method is able to deal even with design of asymmetric FIR filters. In addition, it is noticeable that the main body of the proposed method can not only solve the linear or quadratic semi-infinite programming problems it will also be able to handle other nonlinear programming problems rather than the problems of filter design.

2. PROBLEM STATEMENT

The frequency response of an FIR filter of order N with real coefficients $\xi \in \Re^N$ is characterized by:

$$H(e^{j\omega}) = \Gamma^T(e^{j\omega})\xi \tag{1}$$

where $\Gamma(e^{j\omega})$ is an $N \times 1$ vector with elements:

$$\Gamma_l(e^{j\omega}) = e^{-j\omega l} \qquad , l = 0, 1, ..., N-1 \tag{2}$$

Let $H_d(e^{j\omega})$ denote the desired frequency response and Ω_P and Ω_S are pass-band and stop-band, respectively which are both compact and uncountable subsets of $[0, \pi]$. PCWLSE design problem can be stated as follows:

$$\begin{cases} \min\limits_{\xi \in \Re^N} \|L\|_2 & \text{subject to} \\ \left|H(e^{j\omega}) - H_d(e^{j\omega})\right| < \delta(\omega) & , \omega \in \Omega_{PSd} = \Omega_{Pd} \cup \Omega_{Sd} \end{cases} \tag{3}$$

where

$$\|L\|_2 = w_P \int_{\Omega_P} \left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega + w_S \int_{\Omega_S} \left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega \tag{4}$$

in which w_{P} and w_{S} are nonnegative weighting coefficients for pass-band and stop-band , respectively. Also the stop-band to pass-band weight ratio is chosen as $w_{\text{S}} / w_{\text{P}} >> 1$ [1]. The sets Ω_{Pd} and Ω_{Sd} are finite and countable subsets of Ω_{P} and Ω_{S}, respectively. Here, $\delta(\omega)$ denotes the maximum allowable error in each frequency point. Using the real rotation theorem [12], it can be shown that the problem is converted to the following continuous semi-infinite quadratic programming problem:

$$\begin{cases} \min\limits_{\xi \in \Re^N} \frac{1}{2} \xi^T \psi_r \xi + \varphi^T \xi \quad \text{subject to} \\ \alpha^T(\omega,\lambda)\xi \le \gamma(\omega,\lambda) \ , \omega \in \Omega_{\text{PSd}} = \Omega_{\text{Pd}} \cup \Omega_{\text{Sd}}, \ \lambda \in [0, 2\pi] \end{cases} \tag{5}$$

where

$$\psi_r = \text{Re}\{\psi\} \tag{6}$$

$$\alpha(\omega,\lambda) = \text{Re}\{\Gamma(e^{j\omega}) e^{j\lambda}\} \tag{7}$$

$$\gamma(\omega,\lambda) = \delta(\omega) + \text{Re}\{H_{\text{d}}(e^{j\omega}) e^{j\lambda}\} \tag{8}$$

and $\psi \in \Re^{N \times N}$ is a Toeplitz, Hermitian and positive definite matrix, whose elements $\psi(m,n)$ are defined as:

$$\psi(m,n) = 2\left[w_{\text{P}} \int_{\Omega_{\text{P}}} e^{j(n-m)\omega} d\omega + w_S \int_{\Omega_{\text{S}}} e^{j(n-m)\omega} d\omega \right] \tag{9}$$

and $\varphi \in \Re^N$ is a vector with elements $\varphi(k)$ where:

$$\begin{aligned} \varphi(k) = -2\,\text{Re}\{ & w_{\text{P}} \int_{\Omega_{\text{P}}} H_{\text{d}}(e^{j\omega}) \Gamma_k^*(e^{j\omega}) d\omega \\ & + w_{\text{S}} \int_{\Omega_{\text{S}}} H_{\text{d}}(e^{j\omega}) \Gamma_k^*(e^{j\omega}) d\omega \} \end{aligned} \tag{10}$$

in which $\text{Re}\{.\}$ and $*$ denote the real part and complex conjugate, respectively. The integrals in (9) and (10) can be evaluated using numerical methods such as the Newton-Cotes formula or Simpson's rule. However, for some specific cases as stated below, these integrals are evaluated analytically as follows:

It can be shown that for $\Omega_{\text{P}} = [\omega_{lP}, \omega_{uP}]$ and $\Omega_{\text{S}} = [\omega_{lS}, \omega_{uS}]$ the elements of ψ_r are analytically determined by:

$$\psi_r(m,n) = \begin{cases} \frac{2}{n-m}(w_{\text{P}}(\sin(n-m)\omega_{uP} - \sin(n-m)\omega_{lP}) \\ + w_{\text{S}}(\sin(n-m)\omega_{uS} - \sin(n-m)\omega_{lS})) \quad n \ne m \\ 2(w_{\text{P}}(\omega_{uP} - \omega_{lP}) + w_{\text{S}}(\omega_{uS} - \omega_{lS})) \quad n = m \end{cases} \tag{11}$$

where ω_l and ω_u denote the lower and upper edges of pass-band or stop-band, respectively. Also for a typical desired frequency response stated in (12), the equation in (10) simplifies to (13):

$$H_{\text{d}}(e^{j\omega}) = \begin{cases} e^{-j\tau\omega}, \omega \in [\omega_{lP}, \omega_{uP}] \\ 0, \omega \in [\omega_{lS}, \omega_{uS}] \end{cases} \tag{12}$$

$$\begin{aligned} \varphi(k) = & \frac{-2}{\tau + k}(w_{\text{P}}(\sin(\tau + k)\omega_{uP} - \sin(\tau + k)\omega_{lP}) \\ & + w_{\text{S}}(\sin(\tau + k)\omega_{uS} - \sin(\tau + k)\omega_{lS})) \qquad , k = 0, 1, ..., N-1 \end{aligned} \tag{13}$$

3. PROBLEM SOLUTION

3.1 Changing The Infinite Dimensional Space To Finite Space

Let $\chi = \Omega_{\text{PSd}} \times [0, 2\pi]$ and ρ be a regular Borel measure. The necessary and sufficient Karush-Kuhn-Tucker conditions for the problem defined in (5) can be stated as follows (see [9]):

$$\begin{cases} \psi_r \xi + \varphi + \int_\chi \alpha(\omega,\lambda) d\rho(\omega,\lambda) = 0 \\ \int_\chi (\alpha^T(\omega,\lambda)\xi - \gamma(\omega,\lambda)) d\rho(\omega,\lambda) = 0 \\ \alpha^T(\omega,\lambda)\xi - \gamma(\omega,\lambda) \le 0 \qquad , (\omega,\lambda) \in \chi \\ \rho \ge 0 \end{cases} \tag{14}$$

Let ξ be feasible. Using Caratheodory's theorem, the optimum and non unique measure ρ satisfying (14) can be represented by a measure with finite support (atomic measure) which is not greater than N points [7-9]. Hence the KKT conditions turn into:

$$\begin{cases} \psi_r \xi + \varphi + \sum_{i=1}^{n} \rho_i \alpha(\omega_i, \lambda_i) = 0 \\ \rho_i \ge 0 \qquad , (\omega_i, \lambda_i) \in B \end{cases} \tag{15}$$

where B is the set of binding constraints i.e.

$$B = \{(\omega_r, \lambda_r) \in \chi \,|\, \alpha^T(\omega_r, \lambda_r)\xi = \gamma(\omega_r, \lambda_r)\} \quad , r = 1, 2, ..., n \le N \tag{16}$$

that has changed the infinite dimensional space to a finite space. By the result of Farkas' lema [11] it is concluded that the system of equations defined in (15) can be solved if and only if the system of equations in (17) has no solution.

$$\begin{cases} (\psi_r \xi + \varphi)^T d < 0 \\ \alpha^T(\omega_i, \lambda_i) d \le 0 \qquad for \ (\omega_i, \lambda_i) \in B, \ d \in \Re^N \end{cases} \tag{17}$$

Equations stated in (17) have no solutions if the objective value of the following problem is zero. (For proof see pp. 410-412 of [11]).

$$\begin{cases} \min\limits_{d \in \Re^N} (\psi_r \xi + \varphi)^T d \quad \text{subject to} \\ \alpha^T(\omega_i, \lambda_i) d \le 0 \quad for \ (\omega_i, \lambda_i) \in B \\ -1 \le d_j \le 1 \quad for \ j = 1, 2, ..., N \end{cases} \tag{18}$$

Problem stated in (18) can be solved by conventional Zoutendijk method used for finite space dimensional problems [11], and consequently by the modification of this method called ε-perturbation method which converges to the KKT point of the problem [10].

3.2. Extended ε-Perturbation Approach

ε-perturbation method works with near binding constraints instead of binding constraints. Let $B_k(\varepsilon_k, \xi_k)$ denote near binding constraints corresponding to ξ_k, where ξ_k is the feasible solution at kth iteration and ε_k is a small number used at kth iteration to specify the set of near binding constraints as follows (see [10], [11]):

$$B_k(\xi_k, \varepsilon_k) = \{(\omega_r, \lambda_r) \in \chi \,|\, \alpha^T(\omega_r, \lambda_r)\xi_k - \gamma(\omega_r, \lambda_r) > -\varepsilon_k\} \ , r = 1, 2, ..., p \tag{19}$$

where p is a specific finite number that indicates the number of near binding constraints at each iteration. Since ε_k is very small, usually p is smaller than N. The proposed algorithm has an initialization step and a main step as follows:

Initialization step: The method starts with an initial guess. An initial feasible guess is a vector of filter coefficients ξ_0 that satisfies the constraint of the problem in (5) and is found by solving the following problem:

$$\begin{cases} \min\limits_{(\nu,\xi)\in\Re^+\times\Re^N} \nu \quad \text{subject to} \\ W(\omega)(\alpha^T(\omega,\lambda)\xi - \mathrm{Re}\{H_\mathrm{d}(e^{j\omega})e^{j\lambda}\}) \le \nu \ ,\omega\in\Omega_\mathrm{PSd}, \lambda\in[0,2\pi] \end{cases} \tag{20}$$

where $W(\omega)=1/\delta(\omega)$ and ν denotes a positive real variable. If $\nu_{\min}\le 1$, the problem is feasible, otherwise it is not. This semi-infinite linear programming problem can be solved by use of the simplex extension algorithm [13].
Now, if the problem is feasible set $k=1$ and go to main step.

Main step :
step1: $B_k(\varepsilon_k,\xi_k)$ can be determined by p frequency points $(\omega_r\ ,r=1,2,...,p)$ that satisfy the following equation:

$$\left|H_k(e^{j\omega}) - H_\mathrm{d}(e^{j\omega})\right| > \delta(\omega) - \varepsilon_k \qquad ,\omega\in\Omega_\mathrm{PSd} \tag{21}$$

where $H_k(e^{j\omega}) = \Gamma^T(e^{j\omega})\xi_k$. For the first iteration ε_1 is a small number, typically, 10^{-4}.

Also the corresponding phases (λ_r) are determined by:

$$\lambda_r \approx -\arg\left[H_k(e^{j\omega_r}) - H_\mathrm{d}(e^{j\omega})\right] \tag{22}$$

Let $(z_{k,opt}(\varepsilon_k,\xi_k), d_{k,opt})$ be the optimal solution to the following problem:

$$\begin{cases} \min\limits_{(z,d)\in\Re^{N+1}} z \qquad \text{subject to} \\ (\psi_r\xi_k+\varphi)^T d - z \le 0 \\ \alpha^T(\omega_i,\lambda_i)d - z \le 0 \quad for\ (\omega_i,\lambda_i)\in B_k(\varepsilon_k,\xi_k) \\ -1\le d_j \le 1 \qquad for\ j=1,2,....,N \end{cases} \tag{23}$$

If $z_{k,opt}(\varepsilon_k,\xi_k) \le -\varepsilon_k$ set $\varepsilon_{k+1}=\varepsilon_k$
If $z_{k,opt}(\varepsilon_k,\xi_k) \ge -\varepsilon_k$ set $\varepsilon_{k+1}=0.5\varepsilon_k$
If $z_k(0,\xi_k)=0$, stop and ξ_k is a KKT point and optimal. Otherwise go to step 2.

step2: Solve the following problem:

$$\begin{cases} \min\limits_{\mu\in\Re} \dfrac{1}{2}\xi^T\psi_r\xi + \varphi^T\xi \quad \text{subject to} \\ 0\le\mu\le\mu_{\max} \end{cases} \tag{24}$$

where $\xi=\xi_k+\mu d_k$ and $\mu_{\max}$ is determined by:

$$\mu_{\max} = \begin{cases} \min\limits_{\Phi_k} h(\omega,\lambda) & if\ d_k > \vec{0} \\ \infty & if\ d_k \le \vec{0} \end{cases} \tag{25}$$

in which $\vec{0}$ is an $N\times 1$ null vector and

$$\Phi_k = \left\{(\omega,\lambda)\in\chi - B_k \,\middle|\, \alpha^T(\omega,\lambda)d_k > 0\right\} \tag{26}$$

and also

$$h(\omega,\lambda) = \frac{\gamma(\omega,\lambda) - \alpha^T(\omega,\lambda)\xi_k}{\alpha^T(\omega,\lambda)d_k} \tag{27}$$

Let μ_k be the optimum solution of problem defined in (24). Set $\xi_{k+1}=\xi_k+\mu_k d_k$ and replace $k=k+1$ and go to step1.

Remark 1: Although the approximation used to evaluate λ_r in (22) is good enough even for the few first iterations, it becomes more precise as the algorithm proceeds.
Remark 2: The optimal solution of problem stated in (24) is as follows:

$$\mu_k = \begin{cases} \mu_{\max} & if\ \mu_{\max}\le\mu_1 \\ 0 & if\ \mu_1 \le 0 \\ \mu_1 & if\ 0<\mu_1<\mu_{\max} \end{cases} \tag{28}$$

where

$$\mu_1 = \frac{-(\psi_r\xi_k+\varphi)^T d_k}{d_k^T\psi_r d_k} \tag{29}$$

Remark 3: The minimization in (25) is performed as follows:
Let $h(\omega,\lambda)$ is a function of variable λ , and ω is a constant. We can write (27) as:

$$h(\omega,\lambda) = g(\lambda) = \frac{\delta(\omega) + \mathrm{Re}\{qe^{j\lambda}\} - \mathrm{Re}\{te^{j\lambda}\}}{\mathrm{Re}\{we^{j\lambda}\}} \tag{30}$$

where

$$\begin{cases} q = H_\mathrm{d}(e^{j\omega}) = \mathrm{Re}(H_\mathrm{d}(e^{j\omega})) + \mathrm{Im}(H_\mathrm{d}(e^{j\omega})) = r + js \\ t = \Gamma^T(e^{j\omega})\xi_k = \mathrm{Re}(\Gamma^T(e^{j\omega})\xi_k) + \mathrm{Im}(\Gamma^T(e^{j\omega})\xi_k) = u + jv, \\ w = \Gamma^T(e^{j\omega})d_k = \mathrm{Re}(\Gamma^T(e^{j\omega})d_k) + \mathrm{Im}(\Gamma^T(e^{j\omega})d_k) = x + jy \end{cases} \tag{31}$$

By differentiating of $g(\lambda)$ with respect to λ and using the same technique as in [8] it is easily proved that the minimizer λ in kth iteration, denoted by $\lambda_{\min,k}$ is given by:

$$\lambda_{\min,k} = \sin^{-1}(\theta) - \arg(w) \tag{32}$$

$$\theta = ((u-r)y - (v-s)x)(x^2+y^2)^{-0.5}\,\delta(\omega)^{-1} \tag{33}$$

By Schwarz inequality it is also easily proved that $|\theta|\le 1$. Hence, in equation (32), $\sin^{-1}(\theta)$ is valid (For a simpler case see [8]). Also $\lambda_{\min,k}$ is a function of ω. Thus we can use (32) in (30) to obtain $h(\omega,\lambda_{\min,k}(\omega))$. This is a function of ω and can be minimized easily on $\omega\in\Omega_\mathrm{PSd}$.

4. Numerical Example

We apply the above technique to the following example.

Example: Design an FIR filter with the following specifications:

$$H_{\mathrm{d}}(e^{j\omega}) = \begin{cases} e^{-j16\omega}, \omega \in [0.0.05\pi] \\ 0, \omega \in [0.16\pi, \pi] \end{cases} \quad , \; w_{\mathrm{S}} / w_{\mathrm{p}} = 200$$

$\delta(\omega) \leq 0.01$ in both pass-band and stop-band.

The simulation results for design of an FIR filter of order 35 are illustrated in figures 1-3. All figures are plotted versus frequency. Also the dotted lines specify the constraints of the problem. Figure 1 shows the magnified portion of the pass-band magnitude response. Figure 2 illustrates the overall response and the magnified portion of the magnitude response in the stop-band. The phase response has been shown in figure 3 (a). The corresponding phase error is depicted in figure 3 (b). The designed filter has satisfied all the magnitude constraints while its pass-band phase is almost the same as the desired one. Note that the pass-band phase error is very small.

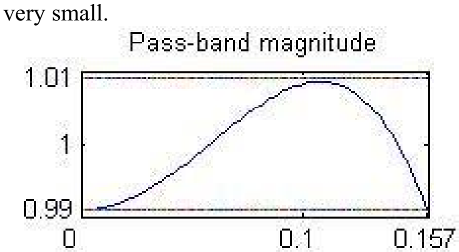


Figure 1. Magnified portion of the magnitude response in the pass-band

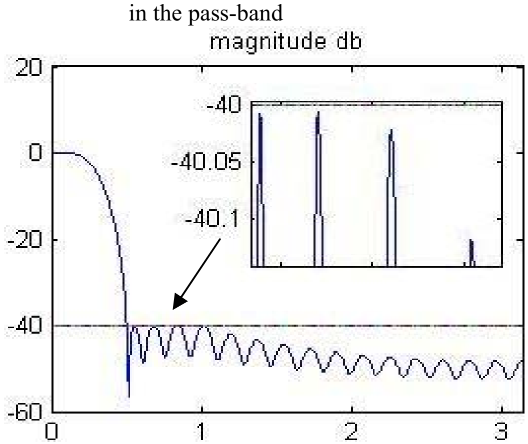
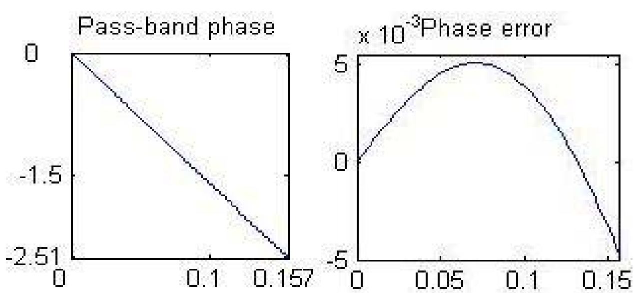


Figure 2. Magnitude response and its corresponding magnified portion in the stop-band.



(a) (b)

Figure 3: (a) Magnified portion of the phase response in the pass-band. (b) Phase error in pass-band.

5. Conclusion

This paper presents a new and easy approach to solve the semi-infinite quadratic programming optimization problem produced by L_2 norm minimization of digital filters while the magnitude of complex Chebyshev error is restricted. Having the advantages of extended active set method, this new approach is very simple to implement and also does not need the calculation of Lagrange multipliers. Calculation of Lagrange multipliers usually needs inversion of some matrixes which are usually ill conditioned and hence the solutions are very sensitive to any small error in data.

The new proposed method is able to deal with design of both symmetric and asymmetric FIR filters. The procedure has been implemented by solving an example on computer. The simulation results show the computational accuracy of the procedure while satisfying the specified restrictions of the problem.

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