COEFFICIENT SYMMETRY FOR IMPLEMENTING ODD-ORDER LAGRANGE-TYPE VARIABLE FRACTIONAL-DELAY FILTERS

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ABSTRACT

Lagrange-type variable fractional-delay (VFD) filters are simple and fundamental tools for high-resolution image interpolation. In this paper, we first reveal and theoretically prove the coefficient-symmetry of odd-order Lagrange-type VFD filters, and then exploit the coefficient-symmetry in implementing the VFD filters. We show that the odd-order Lagrange-type VFD filters can be efficiently implemented as the Farrow structure and even-odd structure, whose subfilters have mostly symmetric or antisymmetric coefficients. Thus, the storage cost for the subfilter coefficients can be reduced by 50%, and the number of multiplications required for VFD filtering can also be reduced by 50%, which facilitates high-speed signal processing.

1. INTRODUCTION

Digital filters with variable fractional group-delay (or phase-delay) are referred to as variable fractional-delay (VFD) digital filters, which have been found useful in various signal processing applications such as comb filter design, digital communications, high-performance speech coding, and sampling rate conversion. Among the developed methods for designing VFD filters [1]-[6], frequency-domain approaches can achieve higher design accuracy than time-domain approaches using interpolating polynomials [1]. However, because polynomial interpolator can be used to derive a simple Lagrange-type VFD FIR filter, which exhibits the maximally flat delay at low frequencies, the Lagrange-type VFD filter is still an attractive candidate for many applications where the digital signal to be delayed (or interpolated) contains relatively low frequency components [1, 2].

In [3], we have proved the coefficient-symmetry of even-order Lagrange-type VFD filters and demonstrated that the coefficient-symmetry can be exploited for efficiently implementing even-order Lagrange-type VFD filters. In this paper, we first reveal and theoretically prove the coefficient-symmetry of odd-order Lagrange-type VFD FIR filters, and then show that the coefficient-symmetry can be efficiently exploited for implementing odd-order Lagrange-type VFD filters as the well-known Farrow structure and the more efficient *even-odd* structure whose subfilters mostly have either symmetric or antisymmetric coefficients. This leads to a 50% reduction of the hardware cost for storing the subfilter coefficients. Furthermore, the number of multiplications required in the VFD filtering process can also be reduced by 50%; thus fast on-line tuning is possible.

2. COEFFICIENT-SYMMETRY

The Lagrange-type VFD filter can be expressed as

$$H(z,p) = \sum_{m=-N_2}^{N_1} c_m(p) z^{-m} \tag{1}$$

with coefficients

$$c_m(p) = \frac{\prod\limits_{i=-N_2, i\neq m}^{N_1} (p-i)}{(-1)^{N_1-m}(N_1-m)!(N_2+m)!} \tag{2}$$

where p denotes the variable fractional-delay, $p \in [-N_2, N_1]$, N_1 and N_2 are positive integers. If $N_1 = N_2$, then the VFD filter order $N = 2N_1$ is even [3], and if $N_1 = N_2 + 1$, then the VFD filter order $N = N_1 + N_2 = 2N_2 + 1$ is odd. Since odd-order Lagrange-type VFD filters are also useful in signal processing applications, it is necessary to develop new coefficient-symmetries for efficiently implementing the odd-order Lagrange-type VFD filters. This section reveals the coefficient-symmetry and provides a rigorous proof.

For N odd, the VFD filter coefficients $c_m(p)$ in (2) can be expressed as normal 1-D polynomials of the VFD parameter p,

$$\begin{aligned} c_m(p) &= \sum_{k=0}^{N} b(m,k)p^k \\ &= \sum_{k=0}^{K} b(m,2k)p^{2k} + \sum_{k=0}^{K} b(m,2k+1)p^{2k+1} \end{aligned} \tag{3}$$

with

$$K = \frac{N-1}{2} = N_2, \qquad -N_2 \le m \le N_1.$$

Therefore, we have the following theorem.

Theorem: The coefficients of the polynomial (3) have the symmetry

$$\begin{cases} b(-m,2k) = b(m,2k), & \text{for } -N_2 \le m \le N_2 \\ b(m,2k) = 0, & \text{for } m = N_1 = N_2 + 1 \\ b(m,0) = \delta(m), & \text{for } -N_2 \le m \le N_1 \end{cases} \tag{4}$$

where $\delta(m)$ is the unit impulse. Furthermore,

$$b(m,N) = -b(-m+1,N), \quad \text{for } -N_2 \le m \le N_1 \tag{5}$$

Proof: see appendix.

3. VFD FILTER IMPLEMENTATION

The coefficient-symmetries (4) and (5) can be exploited for efficiently implementing odd-order Lagrange-type VFD filters as the Farrow structure and even-odd structure as follows. Substituting (25) of the appendix into (1) yields

$$H(z,p) = 1 + \sum_{k=1}^{K} F_k(z)p^{2k} + \sum_{k=1}^{K+1} G_k(z)p^{2k-1} \tag{6}$$

where the subfilters $F_k(z)$ and $G_k(z)$ are defined as

$$\begin{aligned} F_k(z) &= \sum_{m=-N_2}^{N_2} b(m,2k)z^{-m}, \qquad k = 1,2,\cdots,K \\ G_k(z) &= \sum_{m=-N_2}^{N_1} b(m,2k-1)z^{-m}, \qquad k = 1,2,\cdots,(K+1). \end{aligned} \tag{7}$$

Fig. 1 and Fig. 2 illustrate the Farrow structure ($N_2 = 1$) and even-odd structure ($N_2 = 2$). The subfilters have the following interesting properties.

1) The order of $F_k(z)$ is not $(2N_2 + 1)$, but $2N_2$ because
$$b(N_1, 2k) = 0, \qquad \text{for } k = 1,2,\cdots,K$$
which reduces one addition and one multiplication required for computing the output of $F_k(z)$.
2) The subfilters $F_k(z)$ are zero-phase because their coefficients are symmetric, each $F_k(z)$ contains only $(N_2 + 1)$ independent coefficients, and only $(N_2 + 1)$ multiplications are required for obtaining the output. Hence, the number of multiplications is reduced by 50%,
3) The last subfilter $G_{K+1}(z)$ is linear-phase because it has antisymmetric coefficients. Consequently, the number of multiplications required for computing the output of $G_{K+1}(z)$ can be reduced by 50%.

Odd-Order Example

To illustrate the coefficient properties (4) and (5), we consider the case $N_2 = 2$, i.e., $N_1 = 3$, and the order of the Lagrange-type VFD filter is $N = 5$. For simplicity, we write the coefficients $b(m,k)$ in matrix form as

$$\mathbf{B}_o = [b(m,k)]$$

where $-2 \le m \le 3$, and $0 \le k \le 5$. Applying the Matlab function *poly* to (2) gets the coefficient matrix

$$\mathbf{B}_o = \begin{bmatrix} 0 & 0.050 & -0.042 & -0.042 & 0.042 & -0.008 \\ 0 & -0.500 & 0.667 & -0.042 & -0.167 & 0.042 \\ 1 & -0.333 & -1.250 & 0.417 & 0.250 & -0.083 \\ 0 & 1.000 & 0.667 & -0.583 & -0.167 & 0.083 \\ 0 & -0.250 & -0.042 & 0.292 & 0.042 & -0.042 \\ 0 & 0.033 & 0 & -0.042 & 0 & 0.008 \end{bmatrix}.$$

By observing each column of $\mathbf{B}_o$, we know that the coefficient matrix $\mathbf{B}_o$ satisfies the coefficient-symmetry stated in (4) and (5).

4. CONCLUSIONS

In this paper, we have revealed and rigorously proved the coefficient-symmetry of odd-order Lagrange-type VFD filters. Through exploiting the coefficient-symmetry, the odd-order Lagrange-type VFD filters can be efficiently implemented as the Farrow structure and even-structure whose subfilters have mostly either symmetric or antisymmetric coefficients. As a result, the hardware cost for storing the independent subfilter coefficients and the number of multiplications required for computing the subfilter outputs can be reduced by 50%, which facilitates high-speed VFD filtering.

5. APPENDIX
PROOF OF THEOREM

We first consider the case $m = N_1$. If $m = N_1$, then (2) becomes

$$c_m(p) = c_{N_1}(p) = \frac{\prod_{i=-N_2}^{N_2}(p-i)}{(-1)^{N_1-N_1}(N_1-N_1)!(N_2+N_1)!} = \frac{p\prod_{i=1}^{N_2}(p^2-i^2)}{N!}$$

which contains only odd-degree terms like $p, p^3, p^5, \cdots, p^N$. Therefore, it is clear from (3) that

$$b(N_1, 2k) = 0, \qquad \text{for } k = 0,1,2,\cdots,K. \tag{8}$$

Next, let us consider the relation between $b(m,2k)$ and $b(-m,2k)$ for $-N_2 \le m \le N_2$. In this case, because $m \ne N_1$, the expression (2) reduces to

$$c_m(p) = \frac{(p-N_1)\prod_{i=-N_2, i\ne m}^{N_2}(p-i)}{(-1)^{N_1-m}(N_1-m)!(N_2+m)!}. \tag{9}$$

For simplicity, we denote the numerator of (9) as

$$Q_m(p) = (p-N_1)\prod_{i=-N_2, i\ne m}^{N_2}(p-i) \tag{10}$$

and the denominator as

$$\lambda_m = (-1)^{N_1-m}(N_1-m)!(N_2+m)!. \tag{11}$$

Substituting $N_1 = N_2 + 1$ into (11), we can rewrite λ_m as

$$\begin{aligned} \lambda_m &= (-1)^{N_1-m}(N_2+1-m)!(N_2+m)! \\ &= (-1)^{N_1-m}(N_2-m)!(N_2+m)!(N_2+1-m) \\ &= (-1)^{N_1-m}(N_2-m)!(N_2+m)!(N_1-m). \end{aligned} \tag{12}$$

Below, we consider the cases $m = 0$ and $m \ne 0$, but $m \ne N_1$. First, if $m = 0$, then $Q_m(p)$ can be further simplified as

$$Q_m(p) = Q_0(p) = (p-N_1)\prod_{i=1}^{N_2}(p^2-i^2) \tag{13}$$

and λ_m becomes

$$\lambda_m = \lambda_0 = (-1)^{N_1}(N_2!)^2 N_1.$$

Hence,

$$c_0(p) = \frac{Q_0(p)}{\lambda_0} = \frac{(p - N_1)\prod_{i=1}^{N_2}(p^2 - i^2)}{(-1)^{N_1}(N_2!)^2 N_1}. \tag{14}$$

Obviously, $c_0(p)$ contains both even- and odd-degree terms of p (also including a constant term).

If $m \neq 0$, then $Q_m(p)$ in (10) can be rewritten as

$$Q_m(p) = (p - N_1)(p + m) \cdot p \prod_{i=1, i\neq|m|}^{N_2}(p^2 - i^2). \tag{15}$$

Denoting

$$O_m(p) = p \prod_{i=1, i\neq|m|}^{N_2}(p^2 - i^2)$$

we have

$$Q_m(p) = \left[(p^2 - N_1 m) - (N_1 - m)p\right] O_m(p)$$

and thus

$$c_m(p) = \frac{Q_m(p)}{\lambda_m} = \frac{\left[(p^2 - N_1 m) - (N_1 - m)p\right] O_m(p)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!(N_1 - m)}. \tag{16}$$

Since $O_m(p)$ contains only odd-degree terms, $c_m(p)$ can be separated to even- and odd-degree terms as

$$c_m(p) = c_m^{\mathrm{e}}(p) + c_m^{\mathrm{o}}(p) \tag{17}$$

with

$$\begin{aligned} c_m^{\mathrm{e}}(p) &= \frac{-(N_1 - m)p \cdot O_m(p)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!(N_1 - m)} \\ &= \frac{-p \cdot O_m(p)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!} \\ c_m^{\mathrm{o}}(p) &= \frac{(p^2 - N_1 m)O_m(p)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!(N_1 - m)} \end{aligned} \tag{18}$$

where $c_m^{\mathrm{e}}(p)$ and $c_m^{\mathrm{o}}(p)$ represent the even- and odd-degree terms of p in $c_m(p)$, respectively. Similarly, we obtain from (17) and (18) that

$$c_{-m}(p) = c_{-m}^{\mathrm{e}}(p) + c_{-m}^{\mathrm{o}}(p) \tag{19}$$

with

$$\begin{aligned} c_{-m}^{\mathrm{e}}(p) &= \frac{-pO_{-m}(p)}{(-1)^{N_1 + m}(N_2 + m)!(N_2 - m)!} \\ c_{-m}^{\mathrm{o}}(p) &= \frac{(p^2 + N_1 m)O_{-m}(p)}{(-1)^{N_1 + m}(N_2 + m)!(N_2 - m)!(N_1 + m)}. \end{aligned} \tag{20}$$

By considering

$$O_{-m}(p) = p \prod_{i=1, i\neq|-m|}^{N_2}(p^2 - i^2) = p \prod_{i=1, i\neq|m|}^{N_2}(p^2 - i^2) = O_m(p)$$

and

$$(-1)^{N_1 + m} = (-1)^{N_1 - m} \cdot (-1)^{2m} = (-1)^{N_1 - m}$$

it is clear from (18) and (20) that

$$c_m^{\mathrm{e}}(p) = c_{-m}^{\mathrm{e}}(p) = \frac{-p^2 \prod_{i=1, i\neq|m|}^{N_2}(p^2 - i^2)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!} \tag{21}$$

and

$$\begin{aligned} c_m^{\mathrm{o}}(p) &= \frac{p(p^2 - N_1 m) \prod_{i=1, i\neq|m|}^{N_2}(p^2 - i^2)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!(N_1 - m)} \\ c_{-m}^{\mathrm{o}}(p) &= \frac{p(p^2 + N_1 m) \prod_{i=1, i\neq|m|}^{N_2}(p^2 - i^2)}{(-1)^{N_1 - m}(N_2 - m)!(N_2 + m)!(N_1 + m)}. \end{aligned}$$

Equation (21) indicates that $c_m(p)$ and $c_{-m}(p)$ have exactly the same even-degree terms of p, namely,

$$b(-m, 2k) = b(m, 2k), \qquad \text{for } -N_2 \leq m \leq N_2$$

but no constant terms, i.e., in (3),

$$b(m, 0) = 0, \qquad \text{for } m \neq 0. \tag{22}$$

For $m = 0$, we have

$$c_0(p) = \sum_{k=0}^{K} b(0, 2k)p^{2k} + \sum_{k=0}^{K} b(0, 2k+1)p^{2k+1}. \tag{23}$$

Substituting $p = 0$ into (14), we obtain

$$c_0(0) = \frac{(-N_1) \cdot (-1)^{N_2} \prod_{i=1}^{N_2} i^2}{(-1)^{N_1}(N_2!)^2 N_1} = 1.$$

On the other hand, substituting $p = 0$ into (23) leads to

$$c_0(0) = b(0, 0).$$

As a result,

$$b(0, 0) = 1. \tag{24}$$

Combining (22) with (24) and (8) yields

$$b(m, 0) = \delta(m), \quad \text{for } -N_2 \leq m \leq N_1.$$

Hence, the $c_m(p)$ in (3) can be simplified as

$$c_m(p) = \delta(m) + \sum_{k=1}^{K} b(m, 2k)p^{2k} + \sum_{k=0}^{K} b(m, 2k+1)p^{2k+1}. \tag{25}$$

Summarizing the above results yields (4).

The coefficient-symmetry (4) is only for even-degree terms, i.e., the terms including $p^0, p^2, p^4, \cdots, p^{2N_2}$. Next, let us consider the odd-degree terms, i.e., the terms including $p, p^3, p^5, \cdots, p^N$. We start with

$$c_m(p) = \frac{\prod_{i=-N_2, i\neq m}^{N_1} (p-i)}{(-1)^{N_1-m}(N_1-m)!(N_2+m)!} = \frac{Q_m(p)}{\lambda_m} \tag{26}$$

and consider the relation between $c_m(p)$ and $c_{-m+1}(p)$ for $-N_2 \leq m \leq N_1$, from which we can prove the coefficient-symmetry (5) for the Nth degree terms p^N.

Since

$$\begin{aligned} \lambda_{-m+1} &= (-1)^{N_1+m-1}(N_1+m-1)!(N_2-m+1)! \\ &= (-1)^{N_1-m} \cdot (-1)^{2m-1} \cdot (N_2+m)!(N_1-m)! \\ &= -(-1)^{N_1-m}(N_1-m)!(N_2+m)! \\ &= -\lambda_m \end{aligned}$$

we have

$$c_{-m+1}(p) = \frac{Q_{-m+1}(p)}{\lambda_{-m+1}} = \frac{Q_{-m+1}(p)}{-\lambda_m} \tag{27}$$

with

$$Q_{-m+1}(p) = \prod_{i=-N_2, i\neq(-m+1)}^{N_1} (p-i).$$

By performing the substitution

$$i' = -i+1$$

we obtain

$$Q_{-m+1}(p) = \prod_{i'=(N_2+1), i'\neq m}^{-N_1+1} (p+i'-1) = \prod_{i=-N_2, i\neq m}^{N_1} [p+(i-1)].$$

As a result,

$$c_{-m+1}(p) = \frac{\prod_{i=-N_2, i\neq m}^{N_1} [p+(i-1)]}{-\lambda_m}. \tag{28}$$

It is clear from (26) and (28) that a general coefficient-symmetry does not exist for the coefficients of the polynomials $c_m(p)$ and $c_{-m+1}(p)$, but only one exception is the case for the Nth degree terms of p. Since $Q_m(p)$ and $Q_{-m+1}(p)$ have the same coefficient for the term p^N, i.e., the coefficient is 1, thus the coefficients of $c_m(p)$ and $c_{-m+1}(p)$ for p^N are $1/\lambda_m$ and $(-1/\lambda_m)$, respectively. Consequently, we know from (3) that

$$b(m, N) = -b(-m+1, N) = \frac{1}{\lambda_m}$$

which corresponds to (5). This completes the proof of the theorem.

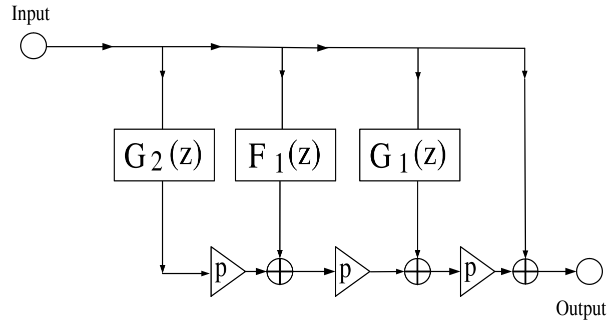

Fig. 1. Farrow structure for odd-order case ($N_2 = 1$).

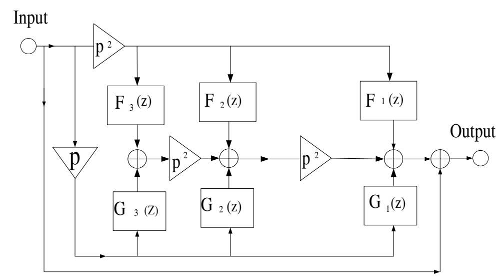

Fig. 2. Even-odd structure for odd-order case ($N_2 = 2$).

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