STUDY OF THE EFFECT OF INTERLEAVERS ON THE POWER SPECTRAL DENSITY APPLICATION TO THE MATRIX INTERLEAVER

W.Chauvet, B.Lacaze, D.Roviras

ENSEEIHT/IRIT/TESA, 2 Rue Camichel, 31071 Toulouse Cedex 7, BP 7122 France
wilfried.chauvet@enseeiht.fr

ABSTRACT

Interleavers used to be considered with a temporal approach. Hence, no work has ever proposed a theoretical study of the frequency effect of an interleaver and we propose here to address this issue. This work is chiefly motivated by the recent use of interleavers as a Spread Spectrum tool. For that purpose, we first point out the equivalence between Periodic Clock Changes (**PCC**) and interleavers. Furthermore, because of its Linear Periodic Time Varying (**LPTV**) nature, a PCC turns any Wide Sense Stationary (**WSS**) process into a Wide Sense Cyclostationary process (**WSC**). Then, we propose to stationarize the output of a PCC in order to derive a theoretical expression of the output Power Spectral Density (**PSD**). Thanks to the previous equivalence, this result can be applied to any interleaver and we propose to investigate the frequency effect of the matrix interleaver.

1. INTRODUCTION

Linear Periodic Time Varying (**LPTV**) filters are characterized by a periodical time varying impulse response [1]. A property of these filters is to turn a Wide Sense Stationary (**WSS**) process into a Wide Sense Cyclostationary (**WSC**) [2] process that can be characterized by a periodical autocorrelation function as well as a bispectrum in the frequency domain [3].

In addition, [4] introduced the definition of Periodical Clock Changes (**PCC**). PCC turn out to constitute a particular subset of the LPTV filters. Furthermore, we will point out in this paper that there is an equivalence between digital PCC and interleavers.

Otherwise, LPTV filters exhibit the property of spectrum spreading when processing an oversampled signal. Hence, recent works [5], [6], [7] propose methods to achieve a Spread Spectrum Multiuser System based on LPTV filters. In [7], a method is proposed to design an orthogonal set of LPTV filters relying on a kernel LPTV filter. In addition, simulations point out that the use of a matrix interleaver as a kernel LPTV filter results in good performance in terms of multiuser interferences. However, in [7], no theoretical consideration was proposed for the spreading efficiency of the matrix interleaver. More generally, given any interleaver, no work has ever theoretically studied the effect of this interleaver on the input Power Spectral Density (**PSD**).

In this paper, we propose to address this issue. As a particular LPTV filter, a PCC turns a WSS process into a WSC process. Relying on results in [3], we propose to stationarize the output of any PCC. This derivation leads to an expression of the stationarized autocorrelation according to the PCC function. Because of the equivalence between PCC and interleavers, this result is of interest since it finally enables to express the effect of any interleaver on the input PSD. Applied to the matrix interleaver, we propose to discuss the influence of the dimension of this interleaver on the spectral effect.

Section 2 is devoted to a review of LPTV filters, PCC and interleavers. In addition, the LPTV nature of PCC as well as the equivalence between digital PCC and interleavers are pointed out. Section 3 proposes the general expression of the stationarized autocorrelation at the output of any PCC. In a spread spectrum framework, section 4 makes the most of this result to discuss the frequency effect of the matrix interleaver. Finally, section 5 presents simulations to validate the theoretical results.

2. LPTV FILTERS, PCC AND INTERLEAVERS

After introducing some definitions, we will point out the LPTV nature of PCC as well as the equivalence between digital PCC and interleavers.

2.1. LPTV filters

An LPTV filter is a filter whose impulse response is periodically time varying [1]. Thus, if we denote $h(n,k)$ as the impulse response of an N periodic LPTV filter, the output $y(n)$ is related to the input $x(n)$ according to relation (1) where $h(n,k)$ satisfies $h(n+N,k) = h(n,k)$.

$$y(n) = \sum_{k=-\infty}^{\infty} h(n,k)x(n-k) \tag{1}$$

2.2. Periodic Clock Changes

An N Periodic Clock Change is defined by an N periodic function $f(n)$ from $\mathbb{Z}$ to $\mathbb{Z}$ such that the output $y(n)$ and input $x(n)$ are related together according to relation (2):

$$y(n) = x(n - f(n)) \quad \text{with } f(n+N) = f(n) \tag{2}$$

Identification of relations (1) and (2) yields the conclusion that an N periodic PCC is a particular case of LPTV filter with $h(n,k) = \delta(k - f(n))$.

2.3. Interleavers

An N periodic interleaver is defined by an interleaver function $\pi(n)$ with the following relation (3) between the output $y(n)$ and the input $x(n)$.

$$y(n) = x(\pi(n)) \quad \text{with } \pi(n+N) = \pi(n) + N \tag{3}$$

With previous definitions, it is clear that an N periodic interleaver is equivalent to an N periodic PCC. Functions $f(n)$ and $\pi(n)$ are therefore related together according to: $\pi(n) = n - f(n)$.

2.4. Illustration: the matrix interleaver

We propose here to illustrate the previous results. We consider the widely used (Q,P) matrix interleaver. Such an interleaver consists in filling a $Q \times P$ matrix by the input signal $x(n)$ column by column. The output signal is obtained by reading the matrix row by row according to figure 1. It is possible to show that the associated PCC function for this interleaver is given by (4) where $n = N\overline{n}^N + \underline{n}_N$ stands for the Euclidian division of n by N. An example of this PCC function is proposed on figure 1-b in the case $(Q,P) = (8,21)$.

$$\begin{cases} f(n) = -(Q-1)c_P(n) + (P-1)b_{P,Q}(n) \\ c_P(n) = \underline{n}_P \quad \text{and} \quad b_{P,Q}(n) = \underline{\overline{n}^P}_Q \end{cases} \tag{4}$$

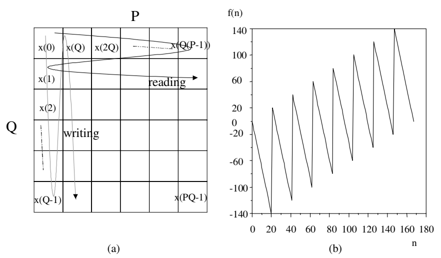


Fig. 1. (a)-Principle of the matrix interleaver (b)-PCC function

3. EFFECT OF A PCC ON THE AUTOCORRELATION

An LPTV filter exhibits the property to turn a WSC process into a WSS process [2]. Furthermore, a WSS process is characterized by a bispectrum whose expression is given in [3]. As a particular LPTV filter, a PCC inherits the same property.

We are interested in deriving the PSD of the process at the output of any PCC. Hence, we propose to derive the autocorrelation function by stationarizing the PCC output process.

3.1. Review of WSS and WSC processes

For a WSS process $y(n)$, we define its autocorrelation function [2] $R_y(p) = E[y(n)y^*(n-p)]$. Stationarity implies that this autocorrelation does not depend on the time index n. The PSD $S_y(f)$ (where f is the normalized frequency) is obtained by computing the Fourier Transform of $R_y(p)$.

If we consider an $L-$WSC process [2] $y(n)$, we can define its autocorrelation function $R_y(m,n) = E[y(m)y^*(n)]$. L cyclostationarity implies an L periodicity in m and n, namely $R_y(m+L, n+L) = R_y(m,n)$. For such an L-WSC process, the bispectrum $S_y(f,f')$ (where f and f' are binormalized frequencies) is obtained from $R_y(m,n)$ by the computation (5) of the two dimensional Fourier Transform [3].

$$\begin{cases} S_y(f',f) = \frac{1}{L}\sum_{q=-\infty}^{\infty} \delta(f-f'+\frac{q}{L})P(f,f') \\ P(f,f') = \sum_{r=0}^{L-1}\sum_{m=-\infty}^{\infty} R_y(m,r)e^{-j2\pi(f'm-rf)} \end{cases} \tag{5}$$

3.2. Stationarization of a WSC process

Let consider an L-WSC process $y(n)$ with an autocorrelation function $R_y(m,n)$. It is possible to stationarize this process [2] and we can show that the stationarized autocorrelation function $R_y^{sta}(p)$ has the following expression (6) according to the WSC autocorrelation.

$$R_y^{sta}(p) = \frac{1}{L}\sum_{r=0}^{L-1} R_y(p+r,r) \tag{6}$$

Proof of this relation is straightforward. Evaluating the bispectrum (5) for $f' = f$ leads to the expression of the PSD $S_y^{sta}(f)$ of the stationarized process. Computing the inverse Fourier Transform of $S_y^{sta}(f)$ yields the result.

3.3. Stationarization of the PCC output

Let consider an N periodic PCC with a PCC function $f(n)$. As a particular case of LPTV filter, this PCC turns a WSS process $x(n)$ with an autocorrelation function $R_x(p)$ into an N-WSC process $y(n)$ with an autocorrelation $R_y(m,n)$.

Using previous results and definitions, we can show that the stationarized autocorrelation $R_y^{sta}(p)$ of the output PCC is given by the relation (7) where the function $\Lambda(r,p)$ has the expression (8) according to the PCC function $f(n)$. The output PSD $S_y^{sta}(f)$ of the stationarized process can be obtained by computing the Fourier Transform of $R_y^{sta}(p)$.

$$R_y^{sta}(p) = \frac{1}{N}\sum_{r=0}^{N-1} R_x(\Lambda(r,p)) \tag{7}$$

$$\Lambda(r,p) = p + f(r) - f(p+r) \tag{8}$$

It is worthy to notice that given $p \in \mathbb{N}$, the function $\Lambda(r,p)$ (for $r \in \{0, 1..N-1\}$) represents the set of distances between the inverse (by the PCC) of all elements that are p-distant after PCC processing. Figure 2 illustrates the meaning of this important function $\Lambda(r,p)$.

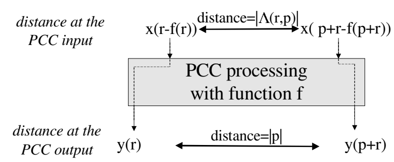


Fig. 2. Illustration of the function $\Lambda(r,p)$

Finally, owing to this relation, given any PCC with function f, it is possible to determine the output stationarized autocorrelation function (and also the PSD) if we know the input process. In addition, the previously stressed equivalence between PCC and interleavers allows to apply this result to any interleaver.

4. APPLICATION TO THE USE OF AN INTERLEAVER IN A SPREAD SPECTRUM FRAMEWORK

LPTV filters exhibit the property to spread the spectrum when processing an oversampled signal. This characteristic makes these filters an attractive tool to achieve a spread sprectrum system. Hence, recent works [6], [7] propose such a system. More precisely, [6] proposes to use a set of random interleavers as an LPTV spreading tool. In such a case, it is shown that the random nature of interleavers implies a perfectly flat PSD at the output of the interleaver. However, in [7], we propose an orthogonal spread spectrum system relying on the matrix interleaver that exhibits good performance in terms of multiuser interference. However, the spreading efficiency of such an interleaver has never been theoretically studied. More generally, no work has ever been led to theoretically study the effect of any interleaver on the input PSD. We aim here at showing how the previously established relation (7) makes possible to tackle this problem through the application to the matrix interleaver.

4.1. Function $\Lambda(r,p)$ for the matrix interleaver

We consider the (Q,P) matrix interleaver previously presented on figure 1 whose period is denoted as $N = PQ$. According to relation (8), we need to express the value of $\Lambda(r,p)$ for $r \in \{0, 1..N-1\}$ and $p \geq 0$ (value for $p < 0$ are useless because of the Hermitian symmetry of the autocorrelation). After derivations detailed in appendix A, we conclude that for a given value of p, the function $\Lambda(r,p)$ can only reach four different values. Given a value of p, these four values can be summed up by the matrix representation of figure 3 where the values of r appear in an increasing way. Depending on the value of p, some of the four zones in the figure 3 can be reduced to an empty zone.

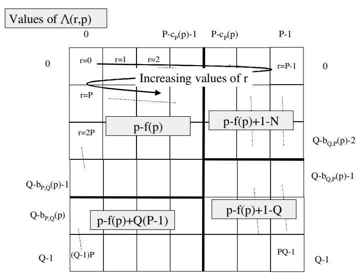


Fig. 3. Possible values of $\Lambda(r,p)$

4.2. Output autocorrelation function

We consider an input WSS process $x(n)$ with an autocorrelation function $R_x(p)$. We make the assumption that there is an integer L fulfilling: $|R_x(p)| = 0$ for $|p| > L$. In addition, we suppose that the parameter Q of the interleaver is chosen greater than L, then we show that the stationary output autocorrelation (7) has only $3L+1$ non null values given by:

$$\left\{ \begin{array}{l} \text{for } j \in \{0, 1..L\}, R_y^{stat}(jP) = R_x(j)\,(1 - \frac{j}{Q}) \\ \text{for } j \in \{0, 1..L-1\}, R_y^{stat}(jP+1) = R_x(j+1)\,\frac{j+1}{N} \\ \text{for } j \in \{1, 2..N\}, R_y^{stat}(N-1-Pj) = R_x(j)\,\frac{(P-1)}{N} j \\ \text{else } R_y^{stat} = 0 \end{array} \right. \quad (9)$$

Now, given an input process $x(n)$ with an autocorrelation $R_x(p)$, we propose to discuss the influence of the parameters P and Q.

4.3. Influence of the parameters P and Q

In relation (9), given a value of P, it is straightforward that $R_y^{stat}(p)$ tends to the function $R^P(p)$ given by (10) for increasing values of Q.

$$R_y^{sta}(p) \to R^P(p) = \left\{ \begin{array}{l} R_x(p/P) \text{ if } \underline{p}_P = 0 \\ 0 \text{ if } \underline{p}_P \neq 0 \end{array} \right. \quad (10)$$

Computing the Fourier Transform of (10), it is possible to show that the output PSD $S_y^{sta}(f)$ tends to the function $S^P(f)$ defined by (11) where $S_x(f)$ stands for the input PSD and f stands for the normalized frequency.

$$S_y^{sta}(f) \to S^P(f) = S_x(Pf) \quad (11)$$

As a consequence, if we consider a matrix interleaver with sufficiently large number of lines, the effect of a matrix interleaver on the PSD of any WSS process is to replicate the compressed input PSD $S_x(f)$ at frequencies multiple of $\frac{1}{P}$. Hence, we perfectly know the spectral effect of the matrix interlaver. We will validate all these results through simulations.

5. SIMULATIONS

We propose here to illustrate and verify the previous results on the matrix interleaver through some simulations.

5.1. Function $\Lambda(r,p)$

We choose the matrix interleaver with parameters $Q = 4$ and $P = 3$ (namely $N = PQ = 12$) and we propose to compute the values of $\Lambda(r,p)$ for $r \in \{0, 1..11\}$ for two different values $p = 2$ and $p = 4$. Figure 4 illustrates the simulation results. These results meet theoretical expectations of the matrix representation on figure 3.

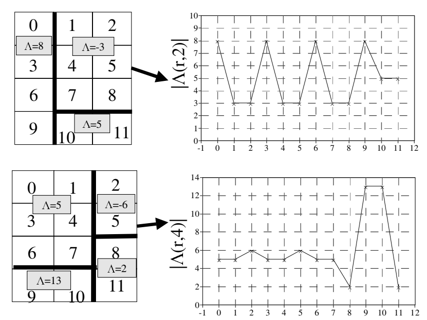


Fig. 4. Function $\Lambda(r,p)$ for $p = 2$ and $p = 4$

5.2. Stationarized autocorrelation function

We propose to illustrate the previous results with an input NRZ process. An NRZ with an oversampled factor L (i.e L samples/symbol) is an $L-WSC$ process whose stationarized autocorrelation function $R_x(m)$ and PSD $S_x(f)$ are given by (12).

$$\left\{ \begin{array}{l} R_x(m) = 1 - \frac{|m|}{L} \text{ if } m \in \{-L..L\} \text{ else } R_x(m) = 0 \\ \text{and } S_x(f) = \frac{1}{L}\left[\frac{\sin(\pi f L)}{\sin(\pi f)}\right]^2 \end{array} \right. \quad (12)$$

Figure 5 illustrates a realization of an NRZ process with an oversampling factor $L = 5$ as well as the autocorrelation function.

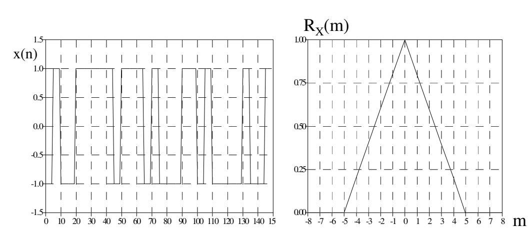
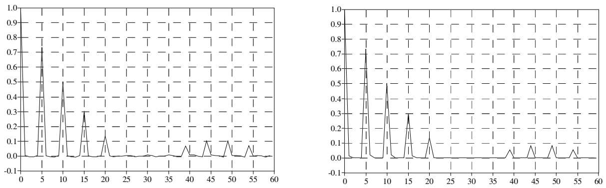


Fig. 5. A realization of an NRZ process and its autocorrelation

Such an NRZ process with oversampling factor L fulfills the previous assumptions of section 4.2. Therefore, the theoretical expressions (9) must be satisfied. On figure 6, we propose to compare these theoretical expressions and the simulated autocorrelation for $(L, P, Q) = (5, 5, 12)$.



(a) (b)

Fig. 6. (a)- simulated (b)- theoretical autocorrelation function

According to this figure 6, simulated results meet the theoretical results.

5.3. PSD after matrix interleaving

According to (11) and (12), if an NRZ process with oversampling factor L is processed by a matrix interleaver, the output PSD for increasing values of Q is given by (13).

$$S_y^{sta}(f) \rightarrow S^P(f) = \frac{1}{L}\left[\frac{\sin(\pi f P L)}{\sin(\pi f P)}\right]^2 \qquad (13)$$

Figure 7 illustrates the PSD of the output process (obtained by computing the Fourier Transform of (9) in case of an NRZ process) for increasing values of Q (namely $Q = 5, 10$ and 60) as well as the theoretical output PSD given by (13). The choice of the other parameters is: $L = 5$ and $P = 6$.

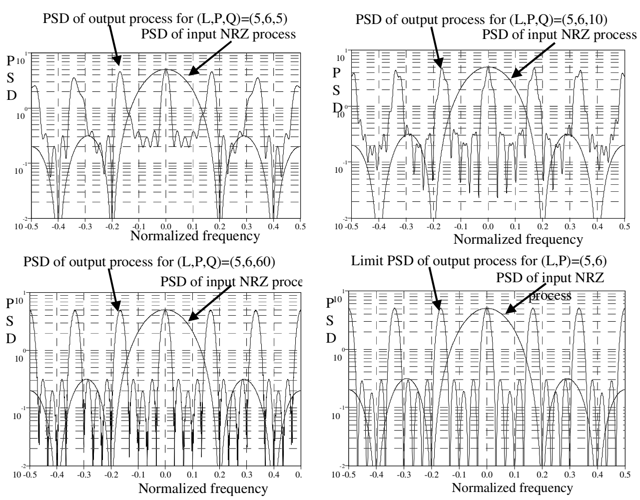


Fig. 7. Output PSD for increasing values of Q

Figure 7 confirms that for a large value of Q, the output PSD is given by P replicas of the compressed input PSD at frequency multiple of $\frac{1}{P}$.

6. CONCLUSION

Recent works propose to use interleavers as a Spread Spectrum tool in a multiuser framework. However, no work has ever proposed a study of the frequency effect of an interleaver. In this paper, we have addressed this issue. By stationarizing the WSC process at the output of a PCC, we derived the general theoretical expression of the stationarized autocorrelation function as well as the PSD. Thanks to the previously stressed equivalence between PCC and interleavers, this result can be applied to any interleaver. We have pointed out how this result is useful through the example of the matrix interleaver. Besides, this widely used interleaver turns out to compress and replicate the input PSD for a large number of columns. Although the results hold for any interleaver, the choice of this particular example was motivated by the use of the matrix interleaver in a recent multiuser system that exhibits good performance in terms of multiuser interference. Furthermore, works are under progress to show that the obtained expressions are very useful to theoretically explain the reasons of these good performance.

APPENDICES

Appendix A: Partial proof of relation (7)

The proof is quite long. Here we give some indications. It consists in using an Euclidian decomposition. Actually, let consider an integer n, and two integers P and Q, there is an unique decomposition of the form (14) where $N = PQ$:

$$\begin{cases} n = A_N(n)N + P b_{P,Q}(n) + c_P(n) \\ A_N(n) = \overline{n}^N \ , c_P(n) = \underline{n}_P \ , b_{P,Q}(n) = \underline{\overline{n}^P}_Q \end{cases} \qquad (14)$$

Using the expression (4) for f, the definition (8) for $\Lambda(r, p)$, as well as the decomposition (14) for $p + r$ and r, it is possible to deduce the following expression for $\Lambda(r, p)$:

$$\begin{aligned} \Lambda(r,p) &= N(1-Q)A_N(p+r) \\ &\quad +(1-N)[b_{P,Q}(p+r) - b_{P,Q}(r)] + Qp \end{aligned}$$

Finally, given an integer p, using this expression for $\Lambda(r, p)$, we discuss the four cases for r (summed up by the 4 zones in the matrix representation of figure 3) and we get the four values of $\Lambda(r, p)$.

Appendix B: Partial proof of relation (9)

Given a value of p in (7), we know that there are only 4 possible values for $\Lambda(r, p)$. Thus, we need to count the number of occurences for these 4 values thanks to the matrix representation depicted on figure 3. Using the additional assumption $|R_x(p)| = 0$ for $|p| < L$ and $Q > L$, we get the result.

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