GENERALIZED SPECTRAL THEORY FOR $\Sigma\Delta$ QUANTIZATION WITH CONSTANT INPUTS

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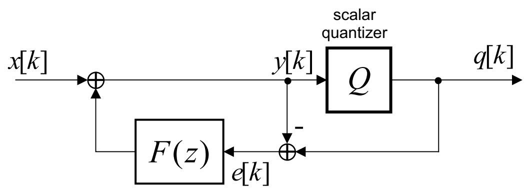

Fig. 1. $\Sigma\Delta$ modulator in its block diagram form of error diffusion.

ABSTRACT

The quantization operation has been traditionally analyzed in signal processing and data conversion as a source of white noise although it is of deterministic nature. This model has become particularly inaccurate in $\Sigma\Delta$ modulation where the quantization resolution can be as low as 1-bit. For an important class of $\Sigma\Delta$ modulators with constant inputs, we build rigorous foundations to the quantization error analysis based on the spectral theory of unitary operators.

1. INTRODUCTION

While amplitude quantization is omnipresent in signal processing, no fundamental signal theory has been built for its genuine analysis as a deterministic operation. When the step size is small, the quantization effect is often and legitimately considered negligible. However, the techniques of data acquisition have evolved towards the use of coarse quantization compensated by oversampling and feedback. In this situation, the only general tool available to the engineers has been the old model of quantization as an additive source of white noise. This model is particularly inaccurate in $\Sigma\Delta$ modulation where resolutions of quantization as low as 1-bit are used [1]. In Figure 1, we recall the principle of a $\Sigma\Delta$ modulator [2], which is similar to the method of error diffusion. The fundamental dif£culty lies in the absence of mathematical tools to obtain an explicit expression of the node signals of a feedback system that is nonlinear due to quantization.

In past research, the instances where a rigorous signal analysis was possible indeed turned out to be in a special case of $\Sigma\Delta$ modulators where a closed form expression of the node signals in terms of the input could be derived. This was the case where the loop £lter satis£es $F(z) = H(z) - 1$ with $H(z) := (1 - z^{-1})^m$ and the quantizer is uniform and not overloaded[1]. This includes the standard single-loop [1] and multi-loop [3] con£gurations, but also the standard multi-stage[2] con£guration [4]. The rigorous spectral analysis of the quantizer error $e[k]$ was then possible by deriving mathematically the Fourier transform $R_e(\omega)$ of the autocorrelation sequence

$$r_e[n] := \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} e[k]e[k+n] \tag{1}$$

de£ned in the time-average sense. In the stationary case of an irrational constant input, it was found the peculiar property that the spectrum of $e[k]$ is purely discrete in the £rst order case $m = 1$, while it becomes white (i.e., $r_e[n] = \sigma_e^2\delta[n]$) as soon as $m \geq 2$.

This past research however did not have any development because the derivations were very speci£c to the considered case of $\Sigma\Delta$ modulation, but also because the techniques of derivation that remained at a straight level of algebraic manipulations did not lead to some new general perspective. One particular dif£culty for potential generalization was the heavy derivation of $r_e[n]$ by means of Fourier expansions to linearize the expressions of $e[k]$ and $e[k+n]$ involved in (1) [1, 4, 3]. In the case of constant inputs, we show in this paper that all these past derivations happen to only uncover particular results in an existing new framework of spectral analysis that applies to a much larger class of $\Sigma\Delta$ modulators. More speci£cally, we consider here all $\Sigma\Delta$ modulators such that the transfer function

$$H(z) := 1 + F(z) \tag{2}$$

takes the form $H(z) = \frac{B(z)}{A(z)}$ with

$$A(z) = 1 + a_1 z^{-1} + \cdots + a_m z^{-m} \quad \text{and} \quad B(z) = (1 - z^{-1})^m \tag{3}$$

and whose quantizer is uniform of any resolution that guarantees the stability of the system and that can be in practice as low as 1-bit [5]. The $\Sigma\Delta$ modulators previously analyzed correspond to the special case where $A(z) = 1$ and the quantizer is of m-bit resolution and will be called the "ideal" modulators. This generalization has been possible thanks to the more recent observation of a tiling phenomenon in these modulators [6] enabling some theoretical closed form expression of the node signals of the feedback system. Thanks to this broader perspective, we £nd that, under an irrational constant input, the autocorrelation sequence $r_e[n]$ appears to result from a sequence of the type $\langle f, \mathcal{U}^n f \rangle_{L^2(\mathbb{T}^m)}$, where $\mathbb{T} := [0, 1)$, $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T}^m)}$ is the inner product of the Hilbert space $L^2(\mathbb{T}^m)$, $\mathcal{U}$ is a speci£c unitary operator of this space and f is an element of $L^2(\mathbb{T}^m)$ which depends on the constant input value

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[1] It is shown in [3] that non-overloading is guaranteed by making the resolution of the quantizer at least m-bit.

[2] In a multi-stage modulator, it can be shown that the error of the quantizer of the last stage is equal to the quantizer error of a multi-bit multi-loop modulator

and the considered $\Sigma\Delta$ con£guration. From the spectral theory of unitary operators, we show that the spectrum of the quantizer error is in general the sum of a purely discrete component and an absolutely continuous component. The particular case considered in prior research (where $A(z) = 1$ and the quantizer is m-bit), is the only case where only one of these two components is present in the spectrum depending of the order, but also the only case where the continuous component is ¤at (white noise component).

This research contains advanced mathematical material and proofs which are developed in details in [7]. The scope of this paper is to convey the new concepts of analysis introduced by this mathematical research to the signal processing engineering community. We will refer to [7] for detailed proofs.

2. BACKGROUND KNOWLEDGE ON $\Sigma\Delta$

2.1. Basics on $\Sigma\Delta$ modulation

As part of basic knowledge on $\Sigma\Delta$ modulation and following the notation of Figure 1, it is easy to show that the global error of the $\Sigma\Delta$ modulator $e_{\Sigma\Delta}[k] := q[k] - x[k]$ and the quantizer error $e[k] := q[k] - y[k]$ by the linear relation

$$e_{\Sigma\Delta}[k] = h[k] * e[k]$$

where $h[k]$ is the sequence whose z-transform is de£ned in (2). In the most typical applications, $x[k]$ is a lowpass signal resulting from oversampling a bandlimited signal, and the goal is to design $H(z)$ as a highpass £lter so that the input-bandwidth portion of the error $e_{\Sigma\Delta}[k]$ is minimized. The case $H(z) = (1 - z^{-1})^m / A(z)$ considered in this paper is important from a theoretical point of view as the m zeros of $H(\omega)$ at $\omega = 0$ allow an mth order asymptotic decay of the input-bandwidth error power with increased oversampling, which constitutes the most basic principle of $\Sigma\Delta$ modulation.

In this paper, we assume that signals are normalized in amplitude so that the step size of the quantizer is 1 and, as it is common usage in $\Sigma\Delta$ modulation, we assume that the quantizer is of "midriser" type, i.e., its output levels belong to $\mathbb{Z} + \frac{1}{2}$.

As it is common practice in $\Sigma\Delta$ modulation we will assume quantizer

2.2. Dynamical system approach

A particularly ef£cient description of the $\Sigma\Delta$ system from a dynamical point of view was derived in [8, 6] and consists of introducing the new signal $u[k]$ whose z-transform is

$$U(z) := \frac{E(z)}{A(z)} = \frac{E_{\Sigma\Delta}(z)}{B(z)}. \tag{4}$$

Any knowledge on the sequence $u[k]$ implies knowledge on the error sequences since $e[k] = a[k] * u[k]$ and $e_{\Sigma\Delta}[k] = b[k] * u[k]$. As one important application, we have

$$r_e[n] = (a[n] * a[-n]) * r_u[n]$$

where $r_u[n]$ is the autocorrelation of $u[k]$, de£ned by

$$r_u[n] := \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} u[k]u[k+n]. \tag{5}$$

Equation (4) implies that $U(z) = (Q(z) - X(z))/((1 - z^{-1})^m$ which means that $u[k]$ is the mth order summation of the $q[k] - x[k]$. By calling in general $u_i[k]$ the ith order summation of the $q[k] - x[k]$, it was shown in [7, 6] that the m-dimensional column vector

$$\mathbf{u}[k] := \big[u_1[k]\ u_2[k]\ \cdots\ u_m[k]\big]^\top,$$

satis£es the recursive relation

$$\mathbf{u}[k] = \mathcal{M}_{x[k]}(\mathbf{u}[k-1]) \tag{6}$$

where for any given real value x, $\mathcal{M}_x$ is the mapping of $\mathbb{R}^m$ de£ned by

$$\mathcal{M}_x(\mathbf{u}) := \mathbf{L}\,\mathbf{u} + \mathbf{i}\,\big(Q(x - \mathbf{f}^\top\,\mathbf{u}) - x\big), \tag{7}$$

$Q(\cdot)$ is the scalar quantizer function,

$$\mathbf{L} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix},\ \ \mathbf{i} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},\ \ \mathbf{f} := \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}, \tag{8}$$

and $f_j := 1 + \sum_{i=n-j+1}^{n} (-1)^{n-j} \binom{i-1}{n-j} a_i$. Solving the recursive relation (6) is the key to evaluating $r_u[n]$ in (5) since at every instant

$$u[k] = p(\mathbf{u}[k]) \tag{9}$$

where $p(\mathbf{u})$ is by de£nition the projection of $\mathbf{u}$ onto its last component.

2.3. Constant input case and error autocorrelation

We now explicitly restrict ourselves to the case of an input $x[k]$ that is equal to a constant x. In this situation, the sequence $\mathbf{u}[k]$ is recursively obtained by

$$\mathbf{u}[k] = \mathcal{M}(\mathbf{u}[k-1]) \tag{10}$$

where $\mathcal{M} := \mathcal{M}_x$ is a £xed mapping independent of k. Since $\mathbf{u}[k+n] = \mathcal{M}^n(\mathbf{u}[k])$ and using (9), one easily £nds from (5) that

$$r_u[n] = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} f_n(\mathbf{u}[k]) \tag{11}$$

where

$$f_n(\mathbf{u}) := p(\mathbf{u}) \cdot p(\mathcal{M}^n(\mathbf{u})). \tag{12}$$

The two obstacles here are that the nth iterate of the nonlinear mapping $\mathcal{M}$ is dif£cult to derive and the evaluation of the discrete sum.

2.4. Outstanding tiling property

When stability is effective, it was observed in [8, 6] the outstanding property that $\mathcal{M}$ has an attracting invariant set Γ that is a $\mathbb{Z}^m$*-tile*. Mathematically, this means that there exists a set $\Gamma \subset \mathbb{R}^m$ such that

(i) $\mathcal{M}(\Gamma) = \Gamma$,

(ii) for any sequence $\mathbf{u}[k]$ that satis£es (10), there exists $n \in \mathbb{Z}$ such that $\mathbf{u}[k] \in \Gamma$ for all $k \geq n$,

(iii) $\{\Gamma + \mathbf{k}\}_{\mathbf{k}\in\mathbb{Z}^m}$ forms a partition of $\mathbb{R}^m$.

This property has been partly demonstrated by experiment [6] and partly proved mathematically [7]. We give a graphical illustration of the tile Γ in the case $m = 2$ in Figure 2(a,c). Qualitatively, the tiling property is mainly due to the fact that $\mathcal{M}_x$ from (7) is of the form

$$\mathcal{M}_x(\mathbf{u}) = \mathbf{L}\,\mathbf{u} - \mathbf{i}\,\bar{x} + \mathcal{K}_x(\mathbf{u}) \tag{13}$$

where $\mathbf{L}$ is a matrix of integer coef£cients and determinant equal to ± 1, $\bar{x} := x + \frac{1}{2}$, and $\mathcal{K}_x$ is a mapping from $\mathbb{R}^m$ to $\mathbb{Z}^m$, which depends on x. As an example, in the single-bit case where $Q(y) = \frac{1}{2}\,\mathrm{sign}(y)$, $\mathcal{K}_x(\mathbf{u}) = \mathbf{i}$ for all $\mathbf{u}$ such that $x - \mathbf{f}^\top\mathbf{u} \geq 0$ and $\mathcal{K}_x(\mathbf{u})$ is equal to the zero vector otherwise.

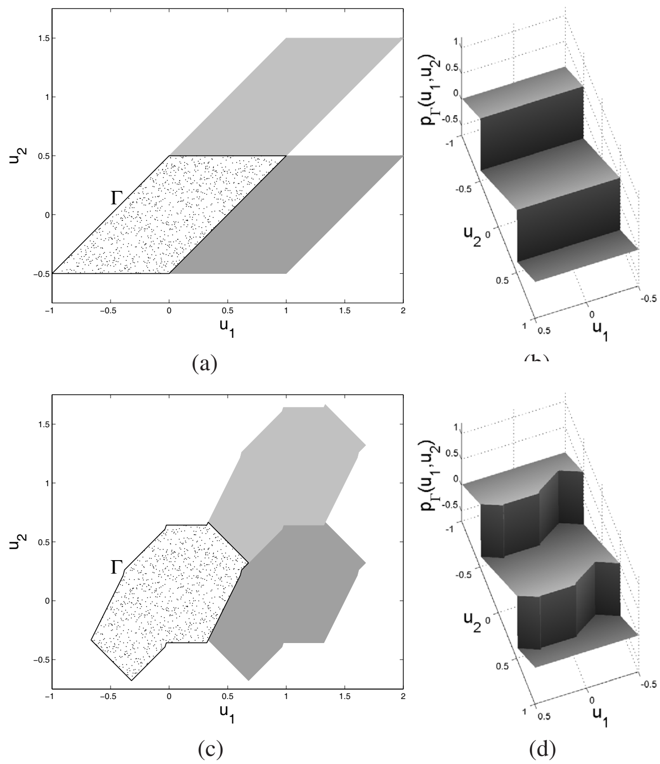

Fig. 2. (a,c) Set Γ outlined in black for two 2nd order modulators with constant input $x = \frac{1}{7}$, experimental sequence $\mathbf{u}[k]$ in black dots (illustrating property (ii)) and representation of $\Gamma + (1,0)$ and $\Gamma + (1,1)$ in gray (illustrating property (iii)) : (a) ideal modulator; (c) $A(z) = 1 - 0.5z^{-1}$ and 1-bit quantizer; (b,d) functions p_Γ corresponding to the cases (a) and (c), respectively.

3. NEW AUTOCORRELATION DERIVATION

3.1. Modulo projections

As explained in [6], given the property that Γ is a tile, there exists a unique function $\langle\cdot\rangle_\Gamma$ of $\mathbb{R}^m$ that is 1-periodic in each dimension (i.e., $\langle\mathbf{u}+\mathbf{k}\rangle_\Gamma = \langle\mathbf{u}\rangle_\Gamma$ for all $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{k} \in \mathbb{Z}^m$) and that is invariant in Γ (i.e., $\langle\mathbf{u}\rangle_\Gamma = \mathbf{u}$ for all $\mathbf{u} \in \Gamma$). Let us de£ne the new mappings

$$\mathcal{L}(\mathbf{u}) := \mathbf{L}\,\mathbf{u} - \mathbf{i}\,\bar{x} \tag{14}$$

$$\mathcal{E}(\mathbf{u}) := \langle\mathcal{L}(\mathbf{u})\rangle_{\mathbb{T}^m} \tag{15}$$

where $\mathbf{i}\,\bar{x}$ results from the expression (13) and $\mathbb{T}^m = [0,1)^m$ is a trivial tile. We have the following property:

Proposition 3.1

$$\mathcal{M} \circ \langle\cdot\rangle_\Gamma = \langle\cdot\rangle_\Gamma \circ \mathcal{E}. \tag{16}$$

Proof: Using the facts that $\langle\mathbf{v}\rangle_T - \mathbf{v}$ always belongs to $\mathbb{Z}^m$ for any tile T and any $\mathbf{v} \in \mathbb{R}^m$, and that $\mathbf{L}$ is only composed of integer coef£cients, one can successively verify that $\mathcal{M}(\langle\mathbf{u}\rangle_\Gamma) - \mathcal{M}(\mathbf{u})$, $\mathcal{M}(\mathbf{u}) - \mathcal{L}(\mathbf{u})$ and $\mathcal{L}(\mathbf{u}) - \mathcal{E}(\mathbf{u})$ belong to $\mathbb{Z}^m$. This implies that $\mathcal{M}(\langle\mathbf{u}\rangle_\Gamma) - \mathcal{E}(\mathbf{u}) \in \mathbb{Z}^m$. Now, because $\mathcal{M}(\langle\mathbf{u}\rangle_\Gamma) \in \Gamma$ (since $\langle\mathbf{u}\rangle_\Gamma \in \Gamma$ and Γ is invariant by $\mathcal{M}$), this automatically implies that $\mathcal{M}(\langle\mathbf{u}\rangle_\Gamma) = \langle\mathcal{E}(\mathbf{u})\rangle_\Gamma$. ■

As a consequence, we have

$$\mathcal{M}^n \circ \langle\cdot\rangle_\Gamma = \langle\cdot\rangle_\Gamma \circ \mathcal{E}^n \tag{17}$$

for any positive integer n. Then,

$$\begin{aligned} f_n(\langle\mathbf{u}\rangle_\Gamma) &= p(\langle\mathbf{u}\rangle_\Gamma) \cdot p(\mathcal{M}^n(\langle\mathbf{u}\rangle_\Gamma)) \\ &= p_\Gamma(\mathbf{u}) \cdot p_\Gamma(\mathcal{E}^n(\mathbf{u})) \end{aligned} \tag{18}$$

where

$$p_\Gamma := p \circ \langle\cdot\rangle_\Gamma. \tag{19}$$

Figure 2 shows graphically two examples of function p_Γ in the case $m = 2$. Note in (18) that contrary to (12), it is the nth iterate of a linear function $\mathcal{E}$ that is needed and which can be derived explicitly. Indeed, $\mathcal{E}^n(\mathbf{u}) = \langle\mathcal{L}^n(\mathbf{u})\rangle_{\mathbb{T}^m}$ and $\mathcal{L}^n(\mathbf{u})$ is obtained by pure linear algebra. Meanwhile, (18) introduced the new function p_Γ which requires some speci£c derivations.

3.2. Ergodicity

When x is an irrational number, $\mathcal{E}$ is known to be ergodic [9]. This results in the following property.

Proposition 3.2 *Let x be an irrational number and Γ be a Lebesgue measurable tile (up to a set of measure zero) that is invariant under $\mathcal{M}$. Then for any function $f \in L^1(\Gamma)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{u}[n]) = \int_\Gamma f(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_{\mathbb{T}^m} f(\langle\mathbf{v}\rangle_\Gamma)\,\mathrm{d}\mathbf{v} \tag{20}$$

for almost every initial condition $\mathbf{u}[0] \in \Gamma$.

As it is standard in the spectral theory of dynamical systems (see, e.g., [9]), let $\mathcal{U} := \mathcal{U}_\mathcal{E}$ be the operator on $L^2(\mathbb{T}^m)$ de£ned by

$$\mathcal{U}g = g \circ \mathcal{E}. \tag{21}$$

By combining (11), (20), (18) and (21), we £nd

$$r_u[n] = \int_{\mathbb{T}^m} p_\Gamma(\mathbf{u}) \cdot \mathcal{U}^n p_\Gamma(\mathbf{u})\mathrm{d}\mathbf{u} \tag{22}$$

$$= \langle p_\Gamma, \mathcal{U}^n p_\Gamma\rangle \tag{23}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of the Hilbert space $L^2(\mathbb{T}^m)$.

4. USE OF SPECTRAL THEORY

The expression (23) shows that properties of the autocorrelation $r_u[n]$ will result from the general analysis of the sequence

$$s_f[n] := \langle f, \mathcal{U}^n f\rangle \tag{24}$$

where $f \in L^2(\mathbb{T}^m)$. The operator $\mathcal{U}$ is easily shown to be unitary (i.e. $\langle\mathcal{U}f, \mathcal{U}g\rangle = \langle f, g\rangle$) from the fact that $\mathcal{E}$ is a mapping of $\mathbb{T}^m$ that preserves measure. Thus, a complete set of knowledge suddenly becomes available thanks to the standard spectral theory of unitary operators [9]. We summarize here the key points useful to our particular problem.

Any function $f \in L^2(\mathbb{T}^m)$ has a unique decomposition

$$f = \bar{f} + \breve{f}, \tag{25}$$

where $\bar{f}$ is the orthogonal projection of f onto the subspace $\mathcal{H}$ spanned by the eigenfunctions of $\mathcal{U}$ and $\breve{f} \in \mathcal{H}^\perp$. Because $\mathcal{U}$ is unitary, $\mathcal{H}$ and $\mathcal{H}^\perp$ are both invariant by $\mathcal{U}$. As a result, one easily derives that

$$s_f[n] = s_{\bar{f}}[n] + s_{\breve{f}}[n]. \tag{26}$$

It is known from spectral theory [9] that the Fourier transform of $s_{\bar{f}}[n]$ is a purely discrete measure and the Fourier transform of $s_{\breve{f}}[n]$ is a continuous measure.

From the particular form of the matrix $\mathbf{L}$ of (8), the space $\mathcal{H}$ is proved in [7] to be

$$\mathcal{H} = \{f \in L^2(\mathbb{T}^m) : f(\mathbf{v}) \text{ only depends on } v_1\}$$

where v_1 is by convention the £rst component of $\mathbf{v}$. As a result, the orthogonal projection of f onto $\mathcal{H}$ is given by

$$\bar{f}(\mathbf{v}) = \int_{\mathbb{T}^{m-1}} f(v_1, \mathbf{v}')\,\mathrm{d}\mathbf{v}'. \tag{27}$$

Remark 4.1 *In the particular case where $m = 1$, $\mathcal{H} = L^2(\mathbb{T})$. Hence $f = \bar{f}$ and $\breve{f}$ is the zero function.*

By abuse of notation, we will write $\bar{f}(\mathbf{v}) = \bar{f}(v_1)$, thus looking at $\bar{f}$ as a function of one variable. We have

$$s_{\bar{f}}[n] = \int_{\mathbb{T}} \bar{f}(v_1)\bar{f}(v_1 + n\bar{x})\,\mathrm{d}v_1 = a_{\bar{f}}(n\bar{x}), \tag{28}$$

where for any 1-periodic function g, $a_g(\tau) := \int_{\mathbb{T}} g(v)g(v+\tau)\mathrm{d}v$ is the autocorrelation of g. If we denote the Fourier coef£cients of $\bar{f}$ by $\hat{\bar{f}}[k]$, then it results from standard Fourier derivations that $s_{\bar{f}}[n] = \sum_k \left|\hat{\bar{f}}[k]\right|^2 e^{2\pi j(k\bar{x})n}$. This leads to the following proposition.

Proposition 4.2 *The spectral component $S_{\bar{f}}(\omega)$ is purely discrete and composed of Dirac peaks of coef£cients $\left|\hat{\bar{f}}[k]\right|^2$ and located at frequencies $2\pi k\bar{x}$ for all $k \neq 0$.*

The computation of $s_{\breve{f}}[n]$ is not easy. However, some fundamental properties on this sequence can still be derived. It is proved in [7] that $S_{\breve{f}}(\omega)$ is non-negative and in L^1. By Riemann-Lebesgue lemma, a fundamental consequence of this property is

$$\lim_{n\to\infty} s_{\breve{f}}[n] = 0. \tag{29}$$

The exact rate at which $s_{\breve{f}}[n]$ goes to zero depends on the function $\breve{f}$. Except in special circumstances $s_{\breve{f}}[n]$ is not reduced to an impulse and thus the spectral component $S_{\breve{f}}(\omega)$ is not white.

5. CONSEQUENCE ON ΣΔ MODULATION

The autocorrelation $r_u[n]$ as expressed in (23) is equal to $s_{p_\Gamma}[n]$ where $s_f[n]$ and p_Γ are de£ned in (24) and (19), respectively. By applying (26), we have

$$r_u[n] = s_{p_\Gamma}[n] = s_{\bar{p}_\Gamma}[n] + s_{\breve{p}_\Gamma}[n]. \tag{30}$$

5.1. Ideal modulators

In ideal modulators, the quantizer is not overloaded, implying that $e[k]$ remains in the interval $[-\frac{1}{2}, \frac{1}{2})$. Moreover, $A(z) = 1$, implying that $u[k] = e[k]$ due to (4). Since $u_m[k] = u[k]$, it is concluded that $p(\Gamma) \subset [-\frac{1}{2}, \frac{1}{2})$. As a result, one can easily prove that

$$p_\Gamma(\mathbf{v}) = \langle p(\mathbf{v})\rangle_0 \tag{31}$$

where we have used here the short notation $\langle\cdot\rangle_0 := \langle\cdot\rangle_{[-\frac{1}{2},\frac{1}{2})}$. This particular function is shown graphically in Figure 2(b) in the case $m = 2$. In the £rst order case $m = 1$, because of Remark 4.1, we simply have $r_u[n] = s_{\bar{p}_\Gamma}[n]$. This immediately implies that $u[k]$ has a purely discrete power spectrum due to Proposition 4.2. This power spectrum is explicitly obtained as follows. Since $p(\mathbf{u}) = \mathbf{u}$, $\bar{p}_\Gamma(\mathbf{u}) = p_\Gamma(\mathbf{u}) = \langle\mathbf{u}\rangle_{[-\frac{1}{2},\frac{1}{2})}$. The Fourier coef£cients of this 1-periodic function can be easily derived to be $\hat{\bar{p}}_\Gamma[k] = j\frac{(-1)^k}{2\pi k}$ for $k \neq 0$ with $\hat{\bar{p}}_\Gamma[0] = 0$. With Proposition 4.2, we conclude that the spectrum of $u[n]$ is solely composed of Dirac peaks of amplitude $\frac{1}{(2\pi k)^2}$ located at frequencies $k\bar{x}$ for $k \neq 0$. This was indeed the result obtained in [1].

Consider now the cases $m \geq 2$. By combining (27) and (31), we £nd that $\bar{p}_\Gamma(\mathbf{v}) = \int_{\mathbb{T}}\langle v\rangle_0 \mathrm{d}v = 0$. We then obtain $r_u[n] = s_{\breve{p}_\Gamma}[n]$, thus implying that the spectrum of $u[n]$ is continuous. Now, with the speci£c function of (31), one actually £nds that $r_u[n] = \frac{1}{12}\delta[n]$ [7] by direct evaluation of the integral (22). Note that $s_{\breve{p}_\Gamma}[n] = \frac{1}{12}\delta[n]$ is a very particular case of (29).

5.2. General modulators

In the general case, the function p_Γ does not have the simple form of (31). Figure 2(d) gives an illustration of function p_Γ in a more generic situation. So in general, $\bar{p}_\Gamma$ is not the zero function and $s_{\breve{p}_\Gamma}[n]$ is not reduced to an impulse. When $m \geq 2$, this implies that the power spectrum of $u[n]$ is a mixture of Dirac components at frequencies $k\bar{x}$ and of a continuous part that is not white. This implies that the spectral analysis previously performed in [1, 3, 4] is actually not representative of the general case and is mainly showing some limit effect.

Further derivations of $r_u[n]$ requires that the set Γ be known. This set is derived in [8] for three particular 2nd order con£gurations. The full parametrization of Γ has been recently achieved by dynamical system analysis on 2nd order modulators within a certain input amplitude range [10].

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