THREE-WAY ARRAYS FOR HARMONIC RETRIEVAL: THE COLORED NOISE CASE

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ABSTRACT

In this work, we deal with multilinear representations of an harmonic process composed by a mixture of complex sinusoids contaminated by a colored Gaussian noise of unknown probability density function. High-Order Statistics, and in particular Fourth-Order Cumulants (FOC), are popular tools to mitigate the effect of the colored noise and can be viewed as a natural enhancement of the noisy mixture. As the FOC of an harmonic process is a multilinear function, it seems natural to map this quantity onto a structured multi-way array (*aka.* tensor). However, an harmonic process can be understood as a pure stationary random process or as a deterministic process corrupted by a stationary noise. In this paper, we explore the relation existing between these two models in the context of the multilinear decomposition theory.

1. INTRODUCTION

Harmonic Retrieval is a very common problem in signal processing, telecommunication (radar, sonar), geophysics, biomedical analysis and power system monitoring. In numerous applications, it is sufficient to assume that the noise is (Gaussian or not) white and thus to employ data-based or second-order statistic for identification of the model. But, it is well-known that if the noise is colored with unknown probability density function (pdf), these approaches are not well adapted. On the other hand, High-Order Statistics (HOS) are a useful tool to combat colored Gaussian noises of unknown pdf. Indeed, the key point of HOS theory is that all Gaussian-noise cumulants of order greater than two are equal to zero [9]. However, researches have essentially focused on model parameter identification for stationary process where multiple realizations are available. More precisely in this case, we have to impose a phase randomization for each harmonic. Consequently, a sum of M harmonics can be seen as a stationary random process for which multiple realizations are available and the Fourth-Order Cumulants (FOC) is well defined by considering ensemble averaging. We call this situation the *random* formulation of the harmonic process. Another direction has been considered in [2] by defining the notion of "mixed" cumulants. The idea behind this term is to consider the harmonic process as a deterministic signal (single realization) and thus to relax the phase randomization assumption. In this case, the non-stationary mixture is the sum of a deterministic process, the model, and a stationary random process, the noise. In this situation, the authors propose a consistent FOC Estimator (FOCE) based on the mixed FOC. We call this situation the *deterministic* formulation of the harmonic process. The first proposal for multi-way array generalizations of factor and principal component analysis date back to the 1960s [1] and early 1970s [5]. The model proposed by Tucker [1] generalized the principal component and factor analysis model in that it used one component matrix for all three "ways" of a three-way data array. These component matrices are related to each other by a so-called core tensor. The model proposed independently by Carroll and Chang and Harshman [5], called the "CANDECOMP" and "PARAFAC" model respectively, also uses component matrices for all three ways, but in their model each component is related to only one component of each of the other two ways. This model is usually called CP for CANDECOMP/PARAFAC. In certain situations, however, this model is too restrictive since it forces to have a super-diagonal core tensor. In such situations, the model proposed by Tucker offers a useful alternative. The above two models can be considered as the fundamental models underlying most currently used multi-way techniques. On the other hand, considering HOS lead to naturally define a mapping between multilinear function onto a multi-way array [3, 4, 7]. Application of multilinear representation to the harmonic process in additive noise has been first addressed by Liu and Sidiropoulos in [8]. More recently, a decomposition of *data* multi-way arrays associated to Shift-Invariant techniques have been exploited [10].

In this work, we revisit the link between the deterministic and the random formulations of the mixture of M harmonic processes in the context of the multilinear algebra of HOS-based multi-way arrays. More precisely, we show that in the *random* formulation, the tensor based on the FOC can be diagonalized as the core tensor of the Tucker model is super-diagonal. In other words, we can express this tensor as a CP model, *ie.*, as a linear combination of M rank-1 tensors (outer product of three Vandermonde vectors). The second part of this work deals with the decomposition of the tensor based on the consistent FOCE in the context of the *deterministic* formulation of the harmonic process. Then, for finite analysis duration, we show that the tensor based on the FOCE follows a Tucker model of order M^3 so we need to consider much more (M^3 in fact) rank-1 tensors to perfectly reconstruct this tensor and its diagonalization is impossible. However, we show that for asymptotic analysis duration the FOCE-based Tucker model tends to the FOC-based CP model of order M and thus the random and the deterministic formulations of an harmonic process become equivalent.

2. TWO TYPES OF HARMONIC PROCESSES

We consider the complex harmonic model defined according to:

$$x_n = \sum_{m=1}^{M} \alpha_m z_m^n, \;\; \text{for} \;\; n \in [0 : N-1] \tag{1}$$

where N is the analysis duration and M is the known number of harmonics, $z_m = e^{i\omega_m}$ is called the m-th pole of x_n where $i = \sqrt{-1}$ and ω_m is the m-th angular-frequency belonging to $(0, \pi]$. In the sequel, we assume that all the angular-frequencies are distinct. In addition, $\alpha_m = a_m e^{i\phi_m}$ is the non-null m-th complex amplitude, *ie.,* $a_m \neq 0, \forall m$. We add a stationary complex colored Gaussian noise e_n in equation (1) and we obtain:

$$y_n = x_n + e_n \text{ where } e_n = h_n * w_n \tag{2}$$

with w_n a Gaussian white noise and h_n is a summable, linear, and time-invariant filter, *ie.* $\sum_{\forall n} |h_n| < \infty$. In addition, we assume that all the moments of the noise exist and are finite. Depending on the following assumptions, model (2) is:

- A pure stationary random harmonic process if ϕ_m's are iid random variable in $[-\pi, \pi)$. In this situation, *pure* means that both x_n and e_n are stationary random process. In this case, we assume that x_n and e_n are two zero-mean statistically independent process.
- A non-stationary Mixed Random harmonic process if ϕ_m's are deterministic in $[-\pi, \pi)$. Here, we use the same terminology as in [2], *ie.*, *mixed* process means that x_n is deterministic and e_n is a stationary random process.

3. FOURTH-ORDER CUMULANTS (FOC) OF A PURE STATIONARY RANDOM HARMONIC PROCESS

3.1. FOC tensor definition

It is well-known that HOS is an efficient way to combat colored Gaussian noise. As the pdf of a pure stationary random harmonic process is symmetric then its Third-Order Cumulant is identically zero (in absence of phase coupling [2]). So, we define the Fourth-Order Cumulant according to:

Definition 1 ([9]) *Assume that $\{y_n,\ n \in [0 : N-1]\}$ is a random stationary zero-mean process defined in (2) which admits finite moments up to the fourth order, the Fourth-Order Cumulant (FOC) is defined according to:*

$$\begin{aligned}[\mathcal{Y}_4]_{\tau_1\tau_2\tau_3} &= \texttt{cum}[y_n y^*_{n+\tau_1} y_{n+\tau_2} y^*_{n+\tau_3}] \\ &= E[y_n y^*_{n+\tau_1} y_{n+\tau_2} y^*_{n+\tau_3}] \\ &- E[y_n y^*_{n+\tau_1}]E[y_{n+\tau_2} y^*_{n+\tau_3}] \\ &- E[y_n y_{n+\tau_2}]E[y^*_{n+\tau_1} y^*_{n+\tau_3}] \\ &- E[y_n y^*_{n+\tau_3}]E[y^*_{n+\tau_1} y_{n+\tau_2}]\end{aligned} \tag{3}$$

where $E[.]$ is the mathematical expectation, $\tau_1 \in [0 : T_1 - 1]$, $\tau_2 \in [0 : T_2 - 1]$, $\tau_3 \in [0 : T_3 - 1]$ and $T_1 + T_2 + T_3 = N + 2$.

Moreover, as the noise and the signal are statistically independent, the FOC tensor of the mixture is given by $\mathcal{Y}_4 = \mathcal{X}_4 + \mathcal{N}$. In other words, the $T_1 \times T_2 \times T_3$ FOC tensor, $\mathcal{Y}_4$, associated to the mixture is simply the sum of the FOC tensor, $\mathcal{X}_4$, associated to a noise-free harmonic process and the FOC tensor, $\mathcal{N}$, associated to the noise.

3.2. FOC tensor of an harmonic process

Taking into account the phase randomization assumption in model (1) and some basic properties of cumulants [9], the FOC of a pure stationary random harmonic process is given by:

$$[\mathcal{X}_4]_{\tau_1\tau_2\tau_3} = -\sum_{m=1}^{M} a_m^4 z_m^{(\tau_1+\tau_3)*} z_m^{\tau_2}. \tag{4}$$

Expression (4) shows that $[\mathcal{X}_4]_{\tau_1\tau_2\tau_3}$ is a sum of 3-way scaled harmonics with null phases [8]. The important point is that expression (4) shares the same poles as model (1) and thus allows the identification of model (1).

3.3. Enhancement of the FOC mixture

The FOC of the noise is given by $\mathcal{N} = \gamma\mathcal{H}$ where $\mathcal{H}$ is the $T_1 \times T_2 \times T_3$ real tensor associated to filter h_n defined by $[\mathcal{H}]_{\tau_1\tau_2\tau_3} = \sum_{n=0}^{N-1} h_n h^*_{n+\tau_1} h_{n+\tau_2} h^*_{n+\tau_3}$ and γ is the Kurtosis "excess" of the driving noise w_n. In case of (colored or not) Gaussian noise, we have a null Kurtosis ($\gamma = 0$) and hence $\mathcal{Y}_4 = \mathcal{X}_4$.

3.4. FOC tensor decomposition

Theorem 1 *The FOC tensor associated to a pure stationary random harmonic process follows a 3-way CP model of order M according to:*

$$\mathcal{Y}_4 = -\sum_{m=1}^{M} a_m^4 \left(p_m^{(1)^*} \circ p_m^{(2)} \circ p_m^{(3)^*}\right) \tag{5}$$

where $\circ$ denotes the outer product and $p_m^{(s)} = \left(1\ z_m\ \ldots z_m^{T_s-1}\right)^T$ is a Vandermonde vector of length T_s.

Corollary 1 *$\mathcal{Y}_4$ associated to a pure stationary random harmonic process is diagonalizeable according to:*

$$\mathcal{Y}_4 = \mathcal{T} \times_1 Z^{(1)^*} \times_2 Z^{(2)} \times_3 Z^{(3)^*} \tag{6}$$

where $\times_s$ is the Tucker's product and we have defined the $T_s \times M$ Vandermonde matrix according to:

$$Z^{(s)} = \begin{bmatrix} p_1^{(s)} & p_2^{(s)} & \ldots & p_M^{(s)} \end{bmatrix} \tag{7}$$

and:

$$[\mathcal{T}]_{jk\ell} = \begin{cases} -a_j^4 & \textit{if } j = k = \ell \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

is an hypercubic $M \times M \times M$ super-diagonal core tensor.

Remark that we assume that $\{Z^{(s)}\}$ are rank-M non-deficient matrices, *ie.*, we have to simultaneously verified for $s \in [1 : 3]$, $M \leq T_s$ and thus $M \leq \min(T_1, T_2, T_3)$. In this case, the CP model is unique up to permutation and scaling of columns [8] if and only if $M \geq 2$.

4. FOURTH-ORDER CUMULANTS ESTIMATOR (FOCE) OF A MIXED PROCESS

In practice when the randomization of the phase is often nonsensical, tensor $\mathcal{Y}_4$ is not "computable" because we have only access to a single realization (of finite length) of the harmonic process defined in (1), *ie.*, for a deterministic value of the phase. As the FOC of a mixed process is non-stationary, we need to an alternative definition. Let y_n be a mixed process in the sense that x_n is a zero-mean deterministic component and e_n is a random component given by equation (2). In this case, we define a new cumulant function, noted $\bar{\texttt{cum}}[.]$, adapted to this mixed situation according to equation

(3) where we replace the true moment by the following mixed moment [2] $\bar{E}[.] = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} E[.]$. Under mild conditions, the mixed process y_n defined in (2) verifies the following property:

$$\lim_{N\to\infty} \hat{\mathcal{Y}}_4^{(N)} \stackrel{\text{def}}{=} \hat{\mathcal{Y}}_4^{(\infty)} \stackrel{\text{w.p.1}}{=} \bar{\mathcal{Y}}_4^{(\infty)} = \bar{\mathcal{X}}_4^{(\infty)} + \gamma\mathcal{H}. \tag{9}$$

where w.p.1 means converge with probability one and symbol $^-$(respectively $\hat{}$) means that we consider mixed (respectively samples) cumulants. This result is well-known for pure stationary random harmonic process and is also valid for mixed processes according to reference [2].

As x_n is the deterministic part of the mixed process y_n, we have $\bar{\mathcal{X}}_4^{(\infty)} = \hat{\mathcal{X}}_4^{(\infty)}$. In addition, as we assume that the noise is Gaussian (colored or not), we have $\gamma = 0$ and expression (9) becomes $\bar{\mathcal{Y}}_4^{(\infty)} = \hat{\mathcal{X}}_4^{(\infty)}$.

4.1. Tucker model of a finite and deterministic harmonic process

According to the FOCE tensor definition, we can formulate the following theorem:

Theorem 2 *Tensor $\hat{\mathcal{X}}_4^{(N)}$ computed from a N-sample harmonic process (with finite N) follows a rank-(M, M, M) Tucker model.*

Proof: The FOCE tensor associated to model (1) admits the following expression:

$$\left[\hat{\mathcal{X}}_4^{(N)}\right]_{\tau_1\tau_2\tau_3} = \sum_{j,k,\ell=1}^{M} \rho_{jk\ell}^{(N)}\, z_j^{*\tau_1} z_k^{\tau_2} z_\ell^{*\tau_3} \tag{10}$$

where:

$$\rho_{jk\ell}^{(N)} = \delta_{jk\ell}^{(N)} - \sum_{s=1}^{3} \delta_{jk\ell}^{(s,N)} \tag{11}$$

and:

$$\delta_{jk\ell}^{(N)} = \frac{\alpha_j^*\alpha_k\alpha_\ell^*}{N} \sum_{m=1}^{M} \alpha_m g_N(z_m z_j^* z_k z_\ell^*), \tag{12}$$

$$\delta_{jk\ell}^{(1,N)} = \frac{\alpha_j^*\alpha_k\alpha_\ell^* g_N(z_k z_\ell^*)}{N^2} \sum_{m=1}^{M} \alpha_m g_N(z_m z_j^*), \tag{13}$$

$$\delta_{jk\ell}^{(2,N)} = \frac{\alpha_j^*\alpha_k\alpha_\ell^* g_N(z_j^* z_\ell^*)}{N^2} \sum_{m=1}^{M} \alpha_m g_N(z_m z_k), \tag{14}$$

$$\delta_{jk\ell}^{(3,N)} = \frac{\alpha_j^*\alpha_k\alpha_\ell^* g_N(z_j^* z_k)}{N^2} \sum_{m=1}^{M} \alpha_m g_N(z_m z_\ell^*). \tag{15}$$

with $g_N(zz') = N$ if $z' = z^*$ and $\frac{1-z^N z'^N}{1-zz'}$ otherwise. As $a_m \neq 0$ and the angular-frequencies are all distinct, $\rho_{jk\ell}^{(N)}$ is a non-zero complex scaling factor. By using the outer product, it comes:

$$\hat{\mathcal{X}}_4^{(N)} = \sum_{j,k,\ell=1}^{M} \rho_{jk\ell}^{(N)} \left(p_j^{(1)^*} \circ p_k^{(2)} \circ p_\ell^{(3)^*}\right). \tag{16}$$

Expression (16) highlights that tensor $\hat{\mathcal{X}}_4^{(N)}$ is a linear combination of M^3 rank-1 structured tensors $p_j^{(1)^*} \circ p_k^{(2)} \circ p_\ell^{(3)^*}$. This is precisely the definition of a rank-(M, M, M) Tucker model. ■

Corollary 2 *For finite duration (fixed N), tensor $\hat{\mathcal{X}}_4^{(N)}$ is non-diagonalizeable.*

Proof: By using the Vandermonde matrices $\{Z^{(s)}\}$ defined in expression (7), expression (16) becomes:

$$\hat{\mathcal{X}}_4^{(N)} = \mathcal{R}^{(N)} \times_1 Z^{(1)^*} \times_2 Z^{(2)} \times_3 Z^{(3)^*} \tag{17}$$

where:

$$\left[\mathcal{R}^{(N)}\right]_{jk\ell} = \rho_{jk\ell}^{(N)} \tag{18}$$

defines the core tensor. Note that $\hat{\mathcal{X}}_4^{(N)}$ is no longer diagonalizeable since the core tensor $\mathcal{R}^{(N)}$ is not super-diagonal (cf. *expression (11)).* ■

4.2. Asymptotic result

An interesting property lies in the relation between the CP model of expression (6) and the Tucker model of expression (17) in the context of infinite analysis duration where relation $\bar{\mathcal{Y}}_4^{(\infty)} = \hat{\mathcal{X}}_4^{(\infty)}$ is valid. More precisely, we formulate the following result:

Theorem 3 *For infinite analysis duration ($N \to \infty$), the Tucker model of order M^3 given in expression (17) reduces to a CP model of order M.*

Proof: By considering the super-diagonal terms in expressions (12), (13) and (15), it comes:

$$\delta_{jjj}^{(N)} = \delta_{jjj}^{(1,N)} = \delta_{jjj}^{(3,N)} = a_j^4 + \frac{a_j^2\alpha_j^*}{N} \sum_{m=1,m\neq j}^{M} \alpha_m g_N(z_m z_j^*), \tag{19}$$

$$\delta_{jj\ell}^{(N)} = \delta_{jj\ell}^{(3,N)} = a_j^2 a_\ell^2 + \frac{a_j^2\alpha_\ell^*}{N} \sum_{m=1,m\neq \ell}^{M} \alpha_m g_N(z_m z_\ell^*), \tag{20}$$

$$\delta_{j\ell\ell}^{(N)} = \delta_{j\ell\ell}^{(1,N)} = a_j^2 a_\ell^2 + \frac{a_\ell^2\alpha_j^*}{N} \sum_{m=1,m\neq j}^{M} \alpha_m g_N(z_m z_j^*). \tag{21}$$

Other terms for $j \neq k \neq \ell$ in expressions (12)-(15) are dominated by N^{-1} or N^{-2} and go to zero when N increases. In particular, we have $\delta_{jk\ell}^{(2,\infty)} = 0$. Now, we can determine the asymptotic behavior of $\rho_{jk\ell}^{(N)}$. We obtain:

$$\rho_{jk\ell}^{(\infty)} = \begin{cases} a_j^4 - a_j^4 - a_j^4 & = & -a_j^4 & \text{for } j = k = \ell, \\ a_j^2 a_\ell^2 - a_j^2 a_\ell^2 & = & 0 & \text{for } j = k \text{ or } \ell, \\ & & 0 & \text{otherwise.} \end{cases} \tag{22}$$

The first (respectively the second) line in result (22) can be obtained by replacing expression (19) (respectively (20) and (21)) in (11). Equivalently, we have:

$$\mathcal{R}^{(\infty)} = \mathcal{T} \tag{23}$$

where tensor $\mathcal{T}$ has been defined in expression (8). Expression (23) is equivalent to $\hat{\mathcal{Y}}_4^{(\infty)} = \mathcal{Y}_4$. ■

It is convenient to be able to represent a tensor as a collection of matrices. Typically, all the columns along a certain mode are rearranged to form a matrix. For instance, we give for $P = 3$, the asymptotic first mode definition of the core tensor:

$$
\mathtt{U}_1(\mathcal{R}^{(\infty)}) = \left[[\mathcal{R}^{(\infty)}]_{:\,:\,0} \quad \ldots \quad [\mathcal{R}^{(\infty)}]_{:\,:\,T_3-1} \right]_{M \times M^2} \tag{24}
$$

where $[\mathcal{R}^{(\infty)}]_{:\,:\,a}$ denotes the a-th vertical slice of tensor $\mathcal{R}^{(\infty)}$.

To illustrate theorem 3, we give for $M = 3$, the asymptotic first mode of the following tensors $[\mathcal{D}^{(\infty)}]_{jkl} = \delta^{(\infty)}_{jkl}$, $[\mathcal{D}^{(\infty)}_s]_{jkl} = \delta^{(s,\infty)}_{jkl}$:

$$
\mathtt{U}_1(\mathcal{D}^{(\infty)}) = \left(\begin{array}{ccc|ccc|ccc} a_1^4 & 0 & 0 & a_1^2a_2^2 & a_1^2a_2^2 & 0 & a_1^2a_3^2 & 0 & a_1^2a_3^2 \\ a_1^2a_2^2 & a_1^2a_2^2 & 0 & 0 & a_2^4 & 0 & 0 & a_2^2a_3^2 & a_2^2a_3^2 \\ a_1^2a_3^2 & 0 & a_1^2a_3^2 & 0 & a_2^2a_3^2 & a_2^2a_3^2 & 0 & 0 & a_3^4 \end{array} \right),
$$

$$
\begin{aligned}
\mathtt{U}_1(\mathcal{D}_1^{(\infty)}) &= \left(\begin{array}{ccc|ccc|ccc} a_1^4 & 0 & 0 & 0 & a_1^2a_2^2 & 0 & 0 & 0 & a_1^2a_3^2 \\ a_1^2a_2^2 & 0 & 0 & 0 & a_2^4 & 0 & 0 & 0 & a_2^2a_3^2 \\ a_1^2a_3^2 & 0 & 0 & 0 & a_2^2a_3^2 & 0 & 0 & 0 & a_3^4 \end{array} \right), \\
\mathtt{U}_1(\mathcal{D}_2^{(\infty)}) &= 0,
\end{aligned}
$$

$$
\mathtt{U}_1(\mathcal{D}_3^{(\infty)}) = \left(\begin{array}{ccc|ccc|ccc} a_1^4 & 0 & 0 & a_1^2a_2^2 & 0 & 0 & a_1^2a_3^2 & 0 & 0 \\ 0 & a_1^2a_2^2 & 0 & 0 & a_2^4 & 0 & 0 & a_2^2a_3^2 & 0 \\ 0 & 0 & a_1^2a_3^2 & 0 & 0 & a_2^2a_3^2 & 0 & 0 & a_3^4 \end{array} \right).
$$

According to above expressions, observe that:

$$
\begin{aligned}
\mathtt{U}_1(\mathcal{R}^{(\infty)}) &= \mathtt{U}_1(\mathcal{D}^{(\infty)}) - \sum_{s=1}^{3} \mathtt{U}_1(\mathcal{D}_s^{(\infty)}) \\
&= \left(\begin{array}{ccc|ccc|ccc} -a_1^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_3^4 \end{array} \right) \\
&= \mathtt{U}_1(\mathcal{T})
\end{aligned}
$$

which is equivalent to result (23). In figure 1, we have reported three typical patterns of convergence for one, two and three harmonics. Note that the convergence is very fast for one component but can be pretty slow for more than one component and for $N > 200$ samples.

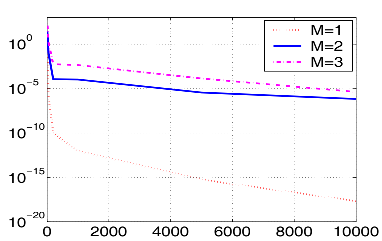

Fig. 1. Typical error values: $||\mathcal{R}^{(N)} - \mathcal{T}||^2$ for one, two and three harmonics Vs. the analysis duration.

Corollary 3 *Theorem 3 indicates that asymptotically, the deterministic and random formulations of the harmonic process are equivalent.*

5. CONCLUSION

In the Harmonic Retrieval problem context, we show that the Fourth-Order Cumulant (FOC) tensor computed from an harmonic process with uniform phase randomization assumption, follows a 3-way CP model of order M. We call this first situation, the random formulation of the harmonic process. If we relax this assumption, the mixture becomes the sum of a deterministic harmonic process and a stationary random Gaussian noise. In this situation, we have to use an alternative definition of the cumulants which is adapted to our new model. This new statistic quantity is called mixed FOC and we show that for finite analysis duration, the consistent FOC Estimator (FOCE) fits a 3-way Tucker model of order M^3. The latter situation, called here deterministic formulation, is very important since the phase randomization assumption is nonsensical in many real applications. However, asymptotically the Tucker model tends to the CP model of lower order and thus the random and the deterministic formulations of the HOS-based harmonic process become equivalent.

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