A STATISTICAL ANALYSIS OF THE DETECTION LIMITS IN FAST PHOTOMETRY

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ABSTRACT

This work investigates the statistical limits for the detection of rapid stellar variations using ground based fast photometry. We show that the noise to be accounted for is of a Mixed Poisson (MP) nature (photon noise mixed by scintillation). As a consequence, three regimes appear for the detection depending on the star's brightness : scintillation; scintillation and photon noise; photon noise and sky background. Both cases of periodic and non periodic signals are investigated. In the case of periodic stellar variations, the minimum amplitude variation that one can detect with a given confidence level is evaluated by analysing the statistics of the MP noise periodogram. The theoretical detection limits are discussed at the light of photometric data obtained on various astronomical sites.

1. INTRODUCTION

The General Combined Catalogue of Variable Stars [1] reports the variability characteristics of tens of class of variable stars. The luminosity variations exhibited by these stars depend on their mass, structure, binarity, evolutionary status, and chemical composition. Their variations may be periodic or quasi-periodic (pulsating stars, rotating variables, eclipsing binaries), or aperiodic (eruptive variables, optical reprocessing in X-ray binaries). Many variables exhibit short timescale, and small amplitude *photometric* variations. Photometric means that we are interested in measuring the quantity of light during a certain amount of time, possibly in a particular band. This paper proposes a statistical evaluation of the performances of ground based photometric observations aimed to detect *rapid stellar variations* with a *high precision*. The typical time scale of the researched oscillations ranges from minutes to hours. Because we are interested in detecting short timescale phenomena, the time sampling of the light curves is typically a few tens of seconds. The technique is called fast photometry. High precision photometry is important because the higher the precision, the stronger the constraint on the physical parameters, and the more accurate the astrophysical models.

The detection limits of stellar variations are known to depend on many parameters. Firstly, on the site : altitude, atmosphere (scintillation, sky transparency variations, extinction in the considered band, light pollution, etc...); secondly, on the telescope (telescope's aperture, reflectivity of primary and secondary mirrors); thirdly, on the position of the target star w.r.t. zenith (airmass); fourthly, on the observation's wavelength and on the filter bandwidth; and finally on some of our world's constants. It is known for a long time how one or a few particular factors above affect the detection [2, 3, 4]. The characterisation of the overall noise in a global statistical setting has however not been carried out yet. In this framework, the contributions attempted in this work are twofold. Firstly, we are interested in modeling analytically the contributions of the noises sources in fast photometric observations, for any given star's magnitude and position in the sky, and for any given observational setting (site & telescope). Secondly, we wish to derive the detection limits imposed by the overall noise. Both the cases of periodic and non periodic stellar variations are investigated (see Sec 2). The results are applied to the sites of Manora Peak (104cm telescope, where fast photometric observations of various objects are currently carried out [4]), to the Devasthal site (where a fully automated telescope is being built, and for which a 3m telescope is planned) and to other astronomical sites (Sec. 4).

The rest of the paper is organised as follows. In the second section we investigate the number of photons to be detected (signal), the statistics of the noise photons, and the Signal-to-Noise Ratio (SNR) in the light curves. The third section turns to the noise level and the SNR in the Fourier amplitude spectrum, and to the detection limits corresponding to a given confidence level. The validity of these results is assessed in Section 4.

2. PHOTONS FROM STARS AND PHOTONS FROM NOISE

2.1. Photons from Stellar Signal

In the following, the subscripts $_*$ and $_{sky}$ correspond respectively to the star and to the sky background. (The sky background is the light coming from the sky, which is never perfectly dark. The corresponding light is thus always detected in addition to that of the star.) The notations F, f and n denote respectively fluxes, magnitudes and numbers of photons. It is customary in Astronomy to use (milli)magnitudes [(m)mag, dimensionless] instead of fluxes [erg/ cm^2/ s /Å]. Evaluating the quantity of light by number of photons is also useful because photons (light quanta) rather fluxes are actually detected. We shall therefore use these three quantities equivalently. The relationship of magnitude to flux is $f = -2.5 \log_{10}[F/F_*(0)]$, where $F_*(0)$ is the reference flux at magnitude zero. The relationship of photons n to flux F is $n = F\Delta\lambda S(\frac{hc}{\lambda})^{-1}$, where $\Delta\lambda$, S, h, c and λ are respectively the filter bandwidth [Å], the surface of the telescope's aperture [cm^2], the Planck's constant [erg s], the speed of light [Å/s] and the wavelength of light [Å]. The quantity $(\frac{hc}{\lambda})$ is then the energy of one photon [erg].

In the case where the stellar variation is not periodic, it will take the form of a random light curve. For the purpose of making a model, we shall quantify this stellar variation by a variation in flux with standard deviation (std) ΔF (or Δf mag, or Δn photons) w.r.t. the average stellar flux. In the case where the variation is periodic, the corresponding std in flux will be denoted by ΔF, or $\Delta f/\Delta n$ as well. We shall focus on the periodic case (pulsating stars typically) because we have in mind to turn to the Fourier spectra later on (Sec. 3). The results of the present Section, which regard light curves, are valid in both the periodic and aperiodic cases : ΔF has just to be given the corresponding definition.

So let us assume that a target star of average magnitude f_* and average flux F_* oscillates with an amplitude (half peak to peak variation) of a few mmag Δf_{max}. Without loss of generality, the oscillation can be assumed to be zero mean and sinusoidal [5]. Then the amplitude variation in mag w.r.t. the average stellar level is $\Delta f_{max} = \sqrt{2}\Delta f$, or, in photons, $\sqrt{2}\Delta n$. With the definitions above, the std in magnitude Δf corresponding to the stellar variation w.r.t. f_* can easily be related to the std in photons Δn by $\Delta f = 2.5\log_{10}(1+\frac{\Delta n}{n_*})$. Using an analysis similar to that of [6], eq. (6), the number of photons from signal falling on the detector during T_{int} seconds is

$$\Delta n = \Delta F \Delta\lambda S\epsilon\, \eta 10^{\{-0.4x(\lambda)\sec Z\}} T_{int}\left(\frac{hc}{\lambda}\right)^{-1} = \Delta F\alpha\left(\frac{hc}{\lambda}\right)^{-1}, \quad (1)$$

where ϵ, η, $x(\lambda)$ and $\sec Z$ are respectively the reflectivity of the mirrors (dimensionless) and possibly the transmission of a Fabry Lens, the quantum efficiency of the detector (dimensionless), the extinction in the considered band [mag] and the airmass (indicating the position of the star w.r.t. zenith, dimensionless). In eq. (1), the coefficient α contains the parameters of the considered observational setting (site & telescope).
The results below regard B band ($\lambda = 4380$Å). A (Johnson) filter of bandwidth 940 Å is assumed. The sky brightness f_{sky} is 22.2 mag in B for both sites, see [7] for a complete list of numerical values. Using these data, the expected number of photons n_* corresponding to F_*, and the number of photons n_{sky} corresponding to the sky background can be derived in the same manner as in eq. (1).
The *signal* that we seek to detect in the light curves is the *stellar variation w.r.t. the average level of light* (star plus sky background). Consequently, the apparent amplitude variation in the light curve $\Delta f_{max,a}$ (w.r.t. star plus background), will be different from Δf_{max} (w.r.t. star only). We obtain

$$\Delta f_{max,a} = 2.5\log(1+\frac{\sqrt{2}\Delta n}{n_* + n_{sky}}) \approx \Delta f_{max}\left(1-\frac{n_{sky}}{n_*}\right). \quad (2)$$

The fainter the stars, the less important the stellar variation relatively to the background. For bright stars, $\Delta f_{max,a}$ is sensibly equal to Δf_{max}; this is not true for faint stars.

2.2. Mixed-Poisson nature of the noise

Now, because of two major random noise effects (scintillation and detection noise), the actual number of detected photons is random, even for a non variable star with constant flux F_*. This noise creates a scatter in the data points of the light curve (with std Δn_{noise} photons), which may hide the stellar variation. This unavoidable scatter should be compared to Δn of eq. (1) to evaluate how much it perturbs the visibility of the stellar signal.
Firstly, random changes in temperature occur in the atmosphere, which in turn generate random fluctuations of the air's refractive index. The effects of these fluctuations are to randomly *defocus* the star, creating scintillation [8]. Consequently, any deterministic flux propagating through these turbulent layers becomes random. The std of the scintillation level, expressed in number of photons Δn_{sc}, can be established from [8], eq. (10), as

$$\Delta n_{sc} = 0.09\, D^{-\frac{2}{3}}(\sec Z)^{1.75} e^{-\frac{h}{h_0}}(2T_{int})^{-\frac{1}{2}} n_* = \beta n_*. \quad (3)$$

In the above equation, D is the telescope diameter (in cm), h the altitude (1951 m for Manora Peak, 2420 m for Devasthal) and $h_0 =$ 8000 m. A typical integration time in fast photometry is $T_{int} = 10$ s. All the results below are established for this value. In eq. (3), the coefficient β contains the parameters of the observational setting.
Secondly, the detection of the photons is random because of the quantum nature of light. Both the star's and sky's lights contribute to the random fluctuations in the detection. In the absence of atmosphere, the statistics of the detected photons for a non variable star with constant flux F_* would be Poissonian, with mean and variance $n_* + n_{sky}$ [2].

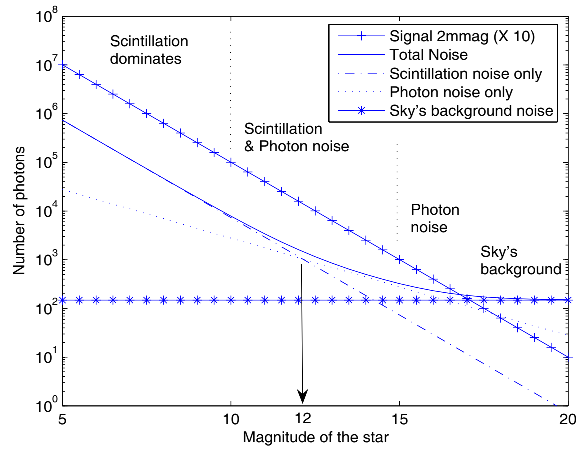

Fig. 1. Number of photons from signal and noise at the Manora Peak site (the signal is magnified by 10 for the sake of visibility).

Accounting now for photon and scintillation noises together, the number of detected photons follows a *doubly stochastic* process : the process is Poisson (because of detection noise) with a mean which is itself stochastic (because of scintillation). The probability of detecting n_{tot} photons given F is $P(n_{tot}|F)$. Let us denote by $p(F)$ the flux's distribution caused by scintillation. From eq. (1), the expected number of detected photons corresponding to a given flux F is $\alpha\frac{\lambda}{hc}F$. But because F is random, we have to account for all the values of F to get the actual distribution of the number of detected photons n_{tot}. The probability of detecting n_{tot} photons is then

$$P(n_{tot}) = \int_0^{+\infty} P(n_{tot}|F)p(F)dF = \int_0^{+\infty}(\frac{\alpha\lambda}{hc}F)^{n_{tot}}\frac{e^{-\frac{\alpha\lambda}{hc}F}}{n_{tot}!}p(F)dF. \quad (4)$$

This defines a *mixed-Poisson* (MP) process. The study of such processes can be found mostly in the literatures of communications (eq. (4) is often referred to as *doubly stochastic processes* [9]), statistical optics (*Poisson-Mandel transform* [10]), actuarial statistics [11], and more recently in astronomy [12]. Depending on the power of the scintillation noise, the mixing process is here either Gaussian, log-normal or follows an F distribution [8]. Now, as far as the noise scatter is concerned, it is sufficient to evaluate the variance of the MP process. Irrespectively of the particular distribution $p(F)$ of the mixing process, the variance of the MP process equals the sum of the variances of the two stochastic processes (photon noise and scintillation in our case) [11]. Consequently, with eq. (3), the associated variance is $n_{sky}+n_*+\beta^2 n_*^2$. The std of the overall noise, expressed in photons and including the joint effects of scintillation and photon (star and sky background) noises, becomes

$$\Delta n_{noise} = \left(n_{sky} + n_* + \beta^2 n_*^2\right)^{\frac{1}{2}}. \quad (5)$$

Indeed, the above expression tends to Δn_{sc} as the photon noise is negligible ($n_{sky} + n_* << \beta^2 n_*^2$), and to $\sqrt{n_{sky}+n_*}$ as the scintillation noise in negligible ($\beta^2 n_*^2 << n_{sky} + n_*$).
Equivalently, the overall scatter becomes in terms of magnitude

$$\Delta f_{noise} = 2.5\log(1+\frac{\sqrt{n_* + n_{sky} + \beta^2 n_*^2}}{n_* + n_{sky}}). \quad (6)$$

The typical noise to be accounted for at Devasthal is presented in Fig. 1 for a 2 mmag amplitude signal (about 0.1% in flux), star at zenith.

Three regimes appear for the detection : for 1 m class telescopes, scintillation dominates for stars brighter than $\approx$ 10th mag; scintillation and photon noises are of comparable influence up to $\approx$ 15th mag stars; photon noise and sky background dominate for fainter stars. From the statistics' viewpoint, the doubly stochastic nature of the noise process is dominated by one of its stochastic component (scintillation) for bright stars. The noise process is clearly doubly stochastic for stars of intermediate magnitude, for which both noises prevail. For fainter stars, it is dominated by the effects of the other stochastic component (photon noise). The magnitude at which photon noise and scintillation are equal is indicated in the graph.
The statistical description above does not appear in the literature dealing with the analysis of photometric light curves. It provides a precise description of how photon and scintillation noise are balanced for any given star's magnitude and observational parameters (through α and β of eq. (1) and (3)). The noise process is described here by the same model (4) irrespectively of the star's magnitude. The two simple situations where either scintillation or photon noise dominates are just two extreme approximations of this model.

2.3. SNR in the Light Curves

The SNR in the temporal domain SNR_t can be evaluated with eq. (1) and (5) (or equivalently with eq. (2) and (6)) as

$$SNR_t = \frac{\Delta n}{\Delta n_{noise}} = \frac{\Delta f_{max,a}/\sqrt{2}}{\Delta f_{noise}} = \frac{\Delta n}{(n_{sky} + n_* + \beta^2 {n_*}^2)^{\frac{1}{2}}}. \tag{7}$$

We show that for 1 m telescopes and for stars brighter than 12th mag, $\approx$ 2 mmag corresponds to the amplitude limit below which the number of signal photons becomes inferior to that of the noise photons ($SNR_t < 1$). This corresponds to a raw detection limit of non periodic signals. As shown below, one can indeed detect much weaker signals if they are periodic, by analysing the Fourier spectrum of the light curves.

3. SPECTRAL ANALYSIS

In the amplitude spectrum, the expected noise level decreases as $1/\sqrt{N}$, where N is the number of data points (see *e.g.* [13]). Using eq. (6), the noise level is here given by

$$\Delta f_{noise/bin} = 2\frac{\Delta f_{noise}}{\sqrt{N}} = \frac{5}{\sqrt{N}} \log\left(1 + \frac{\sqrt{n_* + n_{sky} + \beta^2 n_*^2}}{n_* + n_{sky}}\right). \tag{8}$$

The SNR at signal's frequency (SNR_ν) is the ratio of the expected level of the signal's peak to the expected noise level in the amplitude spectrum. In our case it becomes, see for example [13], eq. (9),

$$SNR_\nu = \frac{\Delta f_{max,a}}{\Delta f_{noise/bin}} = \sqrt{\frac{N}{2}} SNR_t. \tag{9}$$

Numerical evaluations show that the SNR in the signal's frequency bin is fairly high ($>> 1$) for oscillation of a few mmag. This means that in average, the peak of the signal will be SNR_ν times higher than the expected level of the noise. But in each bin, the amplitude spectrum is indeed a random quantity. Consequently, high peaks can be randomly generated by noise. When neither the frequency of the signal nor the SNR are known *a priori*, it may be difficult to reliably detect the signal : many peaks may be signal candidates. Hence, the SNR by itself does not say enough : a confidence level corresponding to a given SNR must be defined and evaluated. In order to do so, it is necessary to determine the statistics of the noise in the spectrum. It is classical for that purpose to assume that the noise is additive, white and Gaussian (*e.g.* [2]). This assumption is questionable, since from Sec. 2, the noise is MP in nature.

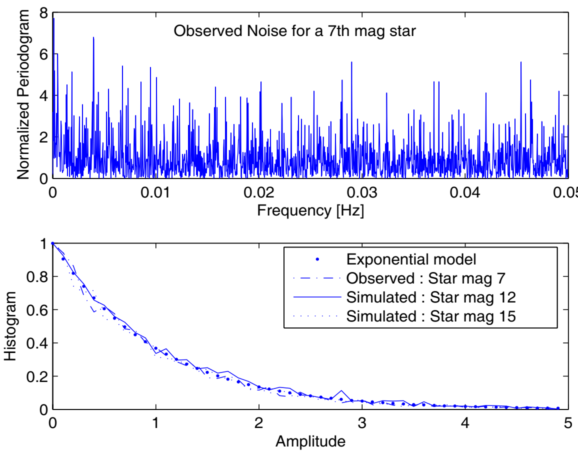

Fig. 2. Statistics of the periodogram. Top panel : Normalised periodogram of the observed noise for the bright star HD 98851. Bottom panel : corresponding histogram of the periodogram, and histograms obtained for mixed-Poisson processes for 12th and 15th mag stars.

In the case of bright stars, the contribution of the photon noise (Poisson component) is negligible (Fig 1). An analysis of the scintillation noise shows that for good astronomical sites, the distribution of the scintillation is actually very close to a Gaussian [7]. So the classical analysis of the periodogram statistics is clearly valid for bright stars. In this case, the periodogram is exponentially distributed [13]. Fig. 2 compares the empirical distribution of the periodogram of the noise obtained for a particular star (HD 98851), to that of a white Gaussian noise. The exponential distribution is clearly a good model.

For fainter stars, the noise process will be Poisson mixed by a Gaussian process, or purely Poisson for very faint stars. In both cases, it is not clear analytically what will be the distribution of the periodogram. We checked this point by simulations for stars of 12th mag (photon noise and scintillation are comparable) and 15th mag (photon noise dominates in the mixed-Poisson process), see Fig. 2. In all cases, the exponential model is accurate. This model is used in the confidence analysis summarized below.
Since the signal's frequency is unknown, one should check how likely is the noise to randomly generate high peaks in the frequency bins. In order to do that, one can firstly compare any given signal peak candidate to the statistics of the peaks' maxima. In the exponential model, the expected level of the maxima in the amplitude spectrum is $M \approx (\sum_{k=1}^{N/2} \frac{1}{k})^{\frac{1}{2}} f_{noise/bin}$. This level gives some insight on the statistical likelihood of high noise peaks. Secondly, a confidence level of C % means that a peak which is SNR_ν times higher than the expected noise level, will not have been generated by the noise C % of the time. C is readily given by $C = 100 \times (1 - e^{-SNR_\nu^2})^{N/2}$. Finally, with the eq. (2) and (8) above, and eq. (18) from [13], we obtain for the minimum amplitude variation f_{min} that can be detected with a given confidence level (Fig. 3)

$$f_{min} = 2.5(1 + \frac{n_{sky}}{n_*}) \log\left(1 + \frac{2\Delta n_{noise}(-\ln[1 - C^{\frac{2}{N}}]^{\frac{1}{2}}}{n_* + n_{sky}}\right). \tag{10}$$

4. COMPARISONS WITH REAL DATA AND DISCUSSIONS

The examples below are drawn from results of the NainiTal-Cape Survey (see [7] for an extensive comparison of the proposed model with real data). The data are time-series photometric observations

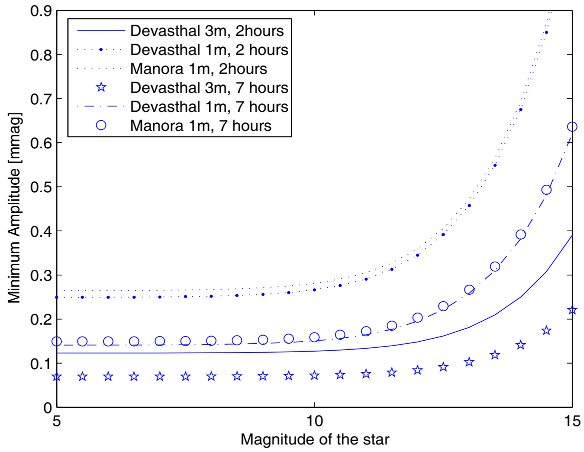

Fig. 3. Minimum amplitude which can be detected with 99% confidence for three observational settings considered in this paper.

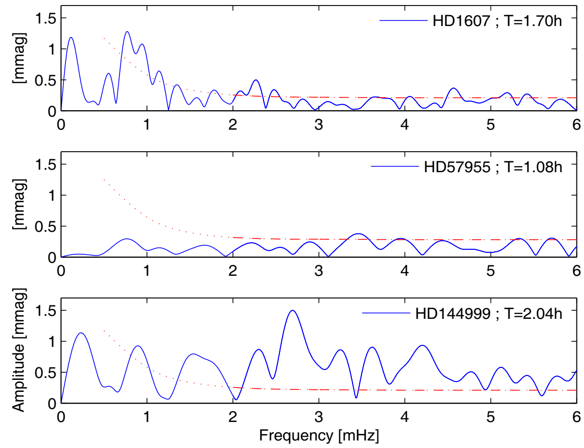

Fig. 4. Examples of spectra obtained at Manora Peak for the NainiTal-Cape Survey. The dashed line indicates the level of the peaks maxima according to the model, for the star's magnitude and the observation duration mentioned in the insets. The dotted line is a schematic representation of the low frequency tail based on the null results from the Survey.

composed of contiguous 10-s integrations lasting for 1 to 3 hr. The data reduction process allows one to remove several atmospheric effects like extinction, sky brithness variations, and sky transparency variations (to some extent).

Regarding the noise scatter in the light curves firstly, the model (eq. (6)) predicts for bright stars (less than 11th mag) and for 1m telescopes to a typical scatter of ≈ 1.1 mmag. The typical scatter actually observed during best nights at Manora Peak is between 1 and 2 mmag. In the example of HD 98851 mentioned above, the scatter is 1.6 mmag. This level is comparable to that observed at Sutherland, South Africa and is typical of other good photometric sites such as La Silla, Canarie Islands, and Granada [7]. For the 3 m telescope at Devasthal, the noise scatter is expected from eq. (6) to reach almost 0.5 mmag for bright stars. For comparison, the authors of [2] reported observations of the scatter for bright stars at Kitt Peak (U.S., 2120 m, 2.1m telescope, average on 8 hours) of about 0.8 mmag with 10 sec integration, and of 0.4 mmag with $T_{int} = 60$ sec. For these settings, our model yields respectively 0.7 and 0.3 mmag.

We shall secondly turn to the noise in the amplitude spectrum. The three examples of Fig. 4 correspond to stars for which no detection could be claimed. Below 2 mHz, residual variations in the sky transparency may create a higher noise level in the spectrum (HD 1607 for example). It does however not affect the detection of rapid variations (above 1 mHz, see [7] for a discussion). For these examples (HD 1607 : $f_* = 9.06$; HD 57955 : $f_* = 8.00$; HD 144999 : $f_* = 8.10$), the predicted noise peak level is 0.24, 0.26 and 0.21 mmag respectively. The actual level of the peaks above 1 mHz is 0.24, 0.28 and 0.78 mmag. So for good nights (the most interesting ones indeed), the MP model (dashed line) evaluates fairly accurately the noise level in the spectrum as well. The third example is representative of a so called *non photometric night*. It is shown to illustrate that the agreement of the model with real data indeed depends on the quality of the night (stability of the atmosphere, etc). Turning finally to a different site, Martinez and Kurtz [3] report an empirical detection limit of 0.35 mmag in 1 h from their observational experience at Sutherland (South Africa), for bright stars and with a 1m telescope (p. 134). For the corresponding parameters and for a 7 mag star, the present model yields a detection limit of 0.36 mmag with 99% confidence.

5. REFERENCES

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