DESIGN OF 2-D FIR FILTERS USING POSITIVE TRIGONOMETRIC POLYNOMIALS

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ABSTRACT

We propose a method for the minimax design of 2-D FIR filters based on a parameterization of multivariate trigonometric polynomials that are positive on a given frequency domain. The parameterization uses sum-of-squares polynomials and so semidefinite programming (SDP) is applicable. The frequency domain is expressed via the positivity of some trigonometric polynomials. The degree of sum-of-square polynomials must be bounded and so the method is in principle suboptimal, but the 2-D FIR filter designs we study numerically suggest that near-optimal results are obtained.

1. INTRODUCTION

Minimax design of 2-D FIR filters is a problem more than 30 years old for which several optimization methods have been proposed. Among them we can cite linear programming [1], multiple exchange [2], iterative weighted least squares [3]. All these methods work on a frequency grid and so they give suboptimal results on the actual passband or stopband, which are compact frequency domains. This paper presents a method based on a new characterization of trigonometric polynomials that are positive on a frequency domain. The method is practically optimal. The only drawback is that the frequency domains (stopband, passband) cannot have arbitrary shape, but have to be defined by the positivity of some trigonometric polynomials.

In the 1-D case, a parameterization of trigonometric polynomials that are positive on an interval $[\alpha, \beta] \subset [0, \pi]$ was given recently in [4], in the form of a linear matrix inequality (LMI). Here, we give a generalization of this result to multivariate polynomials, on frequency domains that are not only cartesian products of intervals (rectangles, in 2-D), but have more general shape.

An overview of the paper is the following. In section 2, multivariate trigonometric polynomials are described and some of their properties are discussed. Section 3 contains the basic result of this paper, namely the parameterization of positive multivariate trigonometric polynomials, in terms of sum-of-squares, and its LMI form. In section 4, we give examples of the 2-D domains that can be used with the parameterization. Sections 5 and 6 describe the algorithm and examples of minimax design of linear phase 2-D FIR filters. Although the method can be implemented only in suboptimal form, due to the necessity of bounding polynomial degrees (that theoretically can be arbitrarily high), the designs appear to be very near the optimum. The notations are fairly standard; in particular, bold letters denote multidimensional entities, e.g. $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d) \in [-\pi, \pi]^d$.

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2. TRIGONOMETRIC POLYNOMIALS

We denote $\boldsymbol{z} = (z_1, \ldots, z_d)$ the d-dimensional complex variable and $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \ldots z_d^{k_d}$ a d-variate monomial of degree $\boldsymbol{k} \in \mathbb{Z}^d$. A symmetric d-variate polynomial of degree $\boldsymbol{n} \in \mathbb{N}^d$ is

$$R(\boldsymbol{z}) = \sum_{\boldsymbol{k}=-\boldsymbol{n}}^{\boldsymbol{n}} r_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}}, \quad r_{-\boldsymbol{k}} = r_{\boldsymbol{k}}^*. \tag{1}$$

On the unit d-circle $\mathbb{T}^d = \{\boldsymbol{z} \in \mathbb{C}^d \mid |z_i| = 1,\ i = 1 : d\}$, with $\boldsymbol{z}^{\boldsymbol{k}} = \exp(j\boldsymbol{k}^T\boldsymbol{\omega})$, the polynomial (1) has real values. A symmetric polynomial $R(\boldsymbol{z})$ with complex coefficients can be written as

$$R(\boldsymbol{z}) = R^e(\boldsymbol{z}) + jR^o(\boldsymbol{z}),$$

where the even part $R^e(\boldsymbol{z})$ and the odd part $R^o(\boldsymbol{z})$ have real coefficients. Symmetric polynomials with real coefficients are named even and denoted with upper index e; note that $R^e(\boldsymbol{z}^{-1}) = R^e(\boldsymbol{z})$.

Any trigonometric polynomial that is *positive* on the unit d-circle (we name it *globally* positive) can be written as a *sum-of-squares* (a proof is e.g. in [5]), namely

$$R(\boldsymbol{z}) = \textstyle\sum_{\ell=1}^{\mu} F_\ell(\boldsymbol{z}) F_\ell^*(\boldsymbol{z}^{-1}), \tag{2}$$

where the polynomials $F_\ell(\boldsymbol{z})$ contain only monomials with nonnegative degree and F_ℓ^* is the polynomial with complex conjugated coefficients. On $\mathbb{T}^d$, this equality becomes

$$R(\mathrm{e}^{j\boldsymbol{\omega}}) \triangleq R(\boldsymbol{\omega}) = \textstyle\sum_{\ell=1}^{\mu} |F_\ell(\boldsymbol{\omega})|^2.$$

Theoretically, the degrees of the polynomials $F_\ell(\boldsymbol{z})$ from (2) can be arbitrarily high. Sum-of-squares trigonometric polynomials can be parameterized in terms of positive semidefinite matrices as follows [6, 7]. The polynomial (1) is sum-of-squares if and only if there exists a matrix $\boldsymbol{Q} \succeq 0$, called Gram matrix associated with $R(\boldsymbol{z})$, such that

$$r_{\boldsymbol{k}} = \mathrm{trace}[\boldsymbol{T}_{\boldsymbol{k}} \cdot \boldsymbol{Q}], \quad \boldsymbol{T}_{\boldsymbol{k}} = \boldsymbol{T}_{k_d} \otimes \ldots \otimes \boldsymbol{T}_{k_1} \tag{3}$$

where $\boldsymbol{T}_{k_i}$ are elementary Toeplitz matrices with ones on the k_i-th diagonal and zeros elsewhere ($\otimes$ denotes the Kronecker product).

In a practical implementation, we have to bound $\deg F_\ell$ to $\boldsymbol{m} \in \mathbb{N}^d$, with $\boldsymbol{m} \geq \boldsymbol{n}$. This condition amounts to taking the matrices $\boldsymbol{T}_{k_i}$ of size $(m_i + 1) \times (m_i + 1)$. Then, the size of the Gram matrix $\boldsymbol{Q}$ is $M \times M$, with $M = \prod_{i=1}^{d}(m_i + 1)$. If the polynomial (1) has complex coefficients, then the matrix $\boldsymbol{Q}$ is complex (and Hermitian); if the polynomial has real coefficients, then $\boldsymbol{Q}$ is real (and symmetric). The parameterization (3) allows the use of SDP in optimization problems involving globally positive trigonometric polynomials, as e.g. in [7, 8].

3. CHARACTERIZATION OF TRIGONOMETRIC POLYNOMIALS POSITIVE ON FREQUENCY DOMAINS

In this section, we present our main contribution, namely the parameterization of trigonometric polynomials that are positive on frequency domains of the form

$$\mathcal{D} = \{\boldsymbol{\omega} \in [-\pi, \pi]^d \mid D_i(\boldsymbol{\omega}) \geq 0, \ i = 1 : \nu\}, \tag{4}$$

where $D_i(\boldsymbol{z})$ are given trigonometric polynomials.

Theorem 1 *If a polynomial $R(\boldsymbol{z})$ defined as in (1) is positive on the nonempty set $\mathcal{D}$ defined in (4) (i.e. $R(\boldsymbol{\omega}) > 0$, $\forall \boldsymbol{\omega} \in \mathcal{D}$), then there exist sum-of-squares polynomials $S_i(\boldsymbol{z})$, $i = 0 : \nu$, such that*

$$R(\boldsymbol{z}) = S_0(\boldsymbol{z}) + \sum_{i=1}^{\nu} D_i(\boldsymbol{z}) \cdot S_i(\boldsymbol{z}). \tag{5}$$

Moreover, if $R(\boldsymbol{z})$ and the polynomials defining $\mathcal{D}$ are even, then the above sum-of-squares polynomials are also even, i.e. we can write

$$R^e(\boldsymbol{z}) = S_0^e(\boldsymbol{z}) + \sum_{i=1}^{\nu} D_i^e(\boldsymbol{z}) \cdot S_i^e(\boldsymbol{z}). \tag{6}$$

The proof is based on results on real polynomials positive on semialgebraic sets [9] and is too long to be presented here.

The reciprocal of Theorem 1 holds in the sense that if the forms (5) or (6) exist, then $R(\boldsymbol{\omega}) \geq 0$, $\forall \boldsymbol{\omega} \in \mathcal{D}$; the proof is trivial. So, some of the polynomials that are nonnegative on $\mathcal{D}$ have the form (5) or (6), but not all of them.

Theoretically, the degrees of the sum-of-squares polynomials from (5) or (6) can be arbitrarily high. In a practical implementation, for complexity reasons, the degrees must be taken as small as possible. Thus, we use the degrees values

$$\begin{aligned} &\deg S_0 = \boldsymbol{n} + \boldsymbol{e}, \\ &\deg S_i = \boldsymbol{n} + \boldsymbol{e} - \deg D_i, \ \ i = 1 : \nu, \end{aligned} \tag{7}$$

where $\boldsymbol{e}$ has small nonnegative elements (preferably $\boldsymbol{e} = \boldsymbol{0}$). In principle, a larger $\boldsymbol{e}$ allows a better approximation by (5) of the set of polynomials (of degree $\boldsymbol{n}$) that are positive on $\mathcal{D}$; however, a larger $\boldsymbol{e}$ means a higher complexity of implementation. With the degree limitation (7), the relation (5) becomes only a sufficient positivity condition. We will see in the experimental section that the impact of this bounded degree in the design applications seems negligible.

Using the Gram matrix parameterization (3) of sum-of-squares polynomials, the relation (5) takes the form of an LMI.

Theorem 2 *If the symmetric polynomial $R(\boldsymbol{z})$ is positive on the domain $\mathcal{D}$ defined as in (4), then there exist matrices $\boldsymbol{Q}_i \succeq 0$ such that*

$$r_{\boldsymbol{k}} = \textit{trace} \left[\boldsymbol{T_k} \boldsymbol{Q}_0 + \sum_{i=1}^{\nu} \boldsymbol{\Psi}_{i\boldsymbol{k}} \boldsymbol{Q}_i \right], \tag{8}$$

where

$$\boldsymbol{\Psi}_{i\boldsymbol{k}} = \sum_{\boldsymbol{\ell}+\boldsymbol{m}=\boldsymbol{k}} (d_i)_{\boldsymbol{\ell}} \boldsymbol{T_m}. \tag{9}$$

Proof. The matrices $\boldsymbol{Q}_i$ are Gram matrices associated with the sum-of-squares $S_i(\boldsymbol{z})$ and so obey to relations similar to (3). The form (9) results immediately by looking at the coefficient of $\boldsymbol{z}^{-\boldsymbol{k}}$ of each product $D_i(\boldsymbol{z}) S_i(\boldsymbol{z})$ from (5). ■

The reciprocal of Theorem 2 holds in the sense that if the matrices $\boldsymbol{Q}_i \succeq 0$, $i = 0 : \nu$, exist such that (8) holds, then $R(\boldsymbol{\omega}) \geq 0$, $\forall \boldsymbol{\omega} \in \mathcal{D}$. (Note that, similarly to the reciprocal of Theorem 1, the strict positivity is replaced by nonnegativity.) The proof is immediate, since (8) is equivalent to (5).

The degree choice (7) imposes the sizes of the Gram matrices $\boldsymbol{Q}_i$. If the polynomials $R(\boldsymbol{z})$ and $D_i(\boldsymbol{z})$ are even (have real coefficients), then the Gram matrices $\boldsymbol{Q}_i$ are real. Otherwise, the Gram matrices are complex.

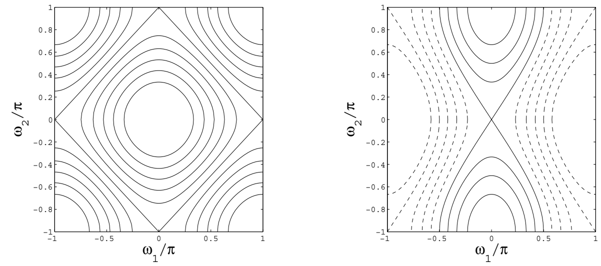

Fig. 1. Left: borders of the domains defined by (11), for $c = -1.5 : 0.3 : 1.5$ (from exterior to interior). Right: borders of the domains defined by (12), for $c = -1.5 : 0.5 : 2.5$

4. FREQUENCY DOMAINS DEFINED BY POSITIVITY OF TRIGONOMETRIC POLYNOMIALS

Usually, the passbands or stopbands of 2-D filters are delimited by simple curves, e.g. circle, ellipse, rhombus ("diamond"), described by low-degree polynomials in $\boldsymbol{\omega}$. Here, we explore what shapes can have frequency domains $\mathcal{D}$ defined by *trigonometric* polynomials, as in (4). We present few basic domains, defined by even trigonometric polynomials. The intersection of several domains is intrinsic to the definition (4); the resulting domain is defined by all the polynomials defining the initial domains. The union of domains is also an allowed operation; since positivity on a domain is described by an LMI, as stated by Theorem 2, it follows that positivity on a union of domains is described by several LMIs, one for each domain.

Rectangles. A rectangle in $[-\pi, \pi]^2$, whose sides are parallel to the axis, is defined by

$$D_1^e(\boldsymbol{\omega}) = \cos\omega_1 - c_1 \geq 0, \qquad D_2^e(\boldsymbol{\omega}) = \cos\omega_2 - c_2 \geq 0. \tag{10}$$

Its complementary is the union of the domains defined by $-D_1^e(\boldsymbol{\omega}) \geq 0$ and $-D_2^e(\boldsymbol{\omega}) \geq 0$, respectively. (The de Morgan rules can be generally applied when working with the complementary of a domain.)

Low band. Another shape suited to describe low frequency bands is defined by

$$D_1^e(\boldsymbol{\omega}) = \cos\omega_1 + \cos\omega_2 - c \geq 0. \tag{11}$$

The curves defined by $D_1^e(\boldsymbol{\omega}) = 0$, representing the borders of the domain defined by (11), are drawn on the left of Figure 1, for several values of the parameter c. For values of c near 2, the shape is almost circular, while for c near 0, it is almost a diamond.

Fan. Shapes suited to fan filters are defined by e.g.

$$D_1^e(\boldsymbol{\omega}) = 2\cos\omega_1 - \cos\omega_2 - c \geq 0 \tag{12}$$

and illustrated on the right of Figure 1, where dashed lines correspond to $c < 1$ and solid lines to $c \geq 1$. It is clear that the coefficient of $\cos\omega_1$ affects the width of the fan on the ω_1 direction.

Diamond. The periodicity of trigonometric polynomials should be taken into account. A diamond shape can be obtained with

$$\begin{aligned} D_1^e(\boldsymbol{\omega}) &= \cos(\omega_1 + \omega_2) - c \geq 0, \\ D_2^e(\boldsymbol{\omega}) &= \cos(\omega_1 - \omega_2) - c \geq 0, \\ D_3^e(\boldsymbol{\omega}) &= \cos\omega_1 + \cos\omega_2 \geq 0. \end{aligned} \tag{13}$$

Since $\omega_1 \pm \omega_2 \in [-2\pi, 2\pi]$, the first two polynomials from (13) define not only the desired diamond, but also four triangles in the corners of $[-\pi, \pi]^2$. The third polynomial from (13) (which has the form (11)) has the purpose of removing these high frequency areas.

5. DESIGN OF LINEAR PHASE FIR FILTERS

We consider here symmetric FIR filters of odd degree, i.e. the simplest (and most common) case of linear phase filters. With no loss of generality, we consider the real coefficients zero-phase filter

$$H^e(\boldsymbol{z}) = \sum_{\boldsymbol{k}=-\boldsymbol{n}}^{\boldsymbol{n}} h_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}}, \quad h_{-\boldsymbol{k}} = h_{\boldsymbol{k}}, \tag{14}$$

which is an even trigonometric polynomial.

The traditional design specifications consist of a passband $\mathcal{D}_p$, on which the frequency response $H^e(\boldsymbol{\omega})$ is ideally equal to 1, a stopband $\mathcal{D}_s$ on which ideally $H^e(\boldsymbol{\omega}) = 0$ and a transition band $\mathcal{D}_t$ on which there is no requirement on $H^e(\boldsymbol{\omega})$. A possible formulation of minimax optimization is to minimize the stopband attenuation γ_s while keeping the passband error below a preset value γ_p. Additionally, we can bound the peak of the frequency response on the transition band. The resulting optimization problem is

$$\begin{array}{ll} \min\limits_{\gamma_s, H^e} & \gamma_s \\ \text{subject to} & |H^e(\boldsymbol{\omega})| \leq \gamma_s, \quad \forall \boldsymbol{\omega} \in \mathcal{D}_s \\ & H^e(\boldsymbol{\omega}) - 1 \leq \gamma_p, \quad \forall \boldsymbol{\omega} \in \mathcal{D}_p \cup \mathcal{D}_t \\ & 1 - H^e(\boldsymbol{\omega}) \leq \gamma_p, \quad \forall \boldsymbol{\omega} \in \mathcal{D}_p \end{array} \tag{15}$$

Furthermore, since the condition $H^e(\boldsymbol{\omega}) - 1 \leq \gamma_p$ is always satisfied in the stopband, we obtain the same solution in (15) by imposing the second constraint on the whole frequency domain $[-\pi, \pi]^2 = \mathcal{D}_p \cup \mathcal{D}_t \cup \mathcal{D}_s$; thus, the implementation is less complex: a globally positive polynomial is characterized by a single Gram matrix, while a polynomial that is positive on a given frequency domain is parameterized by several Gram matrices. Thus, (15) is transformed into an optimization problem with nonnegative polynomials, namely

$$\begin{array}{ll} \min\limits_{\gamma_s, H^e} & \gamma_s \\ \text{s.t.} & R_1^e(\boldsymbol{\omega}) = H^e(\boldsymbol{\omega}) - \gamma_s \geq 0, \quad \forall \boldsymbol{\omega} \in \mathcal{D}_s \\ & R_2^e(\boldsymbol{\omega}) = \gamma_s - H^e(\boldsymbol{\omega}) \geq 0, \quad \forall \boldsymbol{\omega} \in \mathcal{D}_s \\ & R_3^e(\boldsymbol{\omega}) = \gamma_p + 1 - H^e(\boldsymbol{\omega}) \geq 0, \quad \forall \boldsymbol{\omega} \\ & R_4^e(\boldsymbol{\omega}) = H^e(\boldsymbol{\omega}) - 1 + \gamma_p \geq 0, \quad \forall \boldsymbol{\omega} \in \mathcal{D}_p \end{array} \tag{16}$$

We approximate (16), in the sense given by the comments following Theorem 2, by using the parameterization (8) for each of the four positive polynomials from (16). An SDP problem is obtained.

6. 2-D FIR FILTER DESIGN EXAMPLES

We present in this section two examples of minimax design of 2-D FIR filters with real coefficients. These designs should be seen chiefly as benchmarks for the theoretical results, rather than recommendations for practice. In general, it is better to combine minimax and least-squares criteria in the design of FIR filters; our method can be easily adapted to such an approach. All programs have been written in Matlab, using the SDP library SeDuMi [10], and run on a Pentium IV PC at 1GHz.

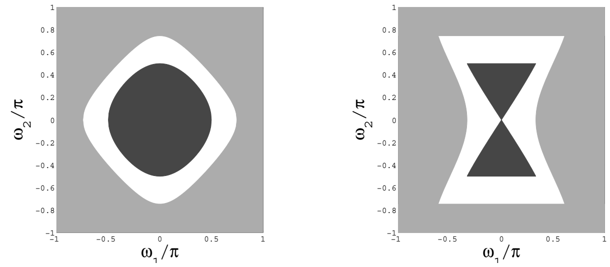

Fig. 2. Passbands (black) and stopbands (gray) for 2-D filter design.

$\boldsymbol{e}$	$\boldsymbol{0}$	(1,1)	(2,2)	(3,3)
Example 1	0.012496	0.012303	0.012297	0.012297
Example 2	0.020837	0.020837	0.020837	-

Table 1. Optimal values of γ_s.

In Figure 2 we present the passband (in black) and stopband (in gray) for our examples: a lowpass and a fan linear phase FIR filters. The filters are designed by solving the SDP form of the problem (16), with some amendments mentioned below. The passband error is $\gamma_p = 0.05$ for the first example and $\gamma_p = 0.1$ for the second. The size of the filters is 15×15, i.e. $\boldsymbol{n} = (7, 7)$ in (14); the degree correction $\boldsymbol{e}$ from (7) is zero, if not otherwise mentioned.

Example 1. The passband and the stopband are defined as in (11), by

$$\begin{aligned} \mathcal{D}_{p1} &= \{\omega_1, \omega_2 \in [-\pi, \pi] \mid \cos\omega_1 + \cos\omega_2 - 1 \geq 0\}, \\ \mathcal{D}_{s1} &= \{\omega_1, \omega_2 \in [-\pi, \pi] \mid -\cos\omega_1 - \cos\omega_2 + 0.3 \geq 0\}. \end{aligned}$$

This is the simplest possible case, as each band is described by a single polynomial. The frequency response of the filter is shown in Figure 3. The design time is about 40 seconds. The optimal value of the stopband error reported by the SDP program is $\gamma_s = 0.012496$.

To check the effect of the degree bound (7), we consider nonzero values of the correction $\boldsymbol{e}$, for which we rerun the SDP program. (When e.g. $\boldsymbol{e} = (2, 2)$, we solve (16) with $\boldsymbol{n} = (9, 9)$, but force to zero all filter coefficients corresponding to monomials $z_1^{k_1} z_2^{k_2}$ with either $|k_1| > 7$ or $|k_2| > 7$.) The optimal values of the stopband error γ_s are shown in Table 1. We see that taking $\boldsymbol{e} = (1, 1)$ improves the error to $\gamma_s = 0.012303$, but further increase of the degree has almost no effect. The frequency response obtained with $\boldsymbol{e} = (1, 1)$ is given in Figure 4; the equiripple character is more evident than in Figure 3, where there are some small irregularities in the high frequency area. We note from Table 1 that, for the other example, the best γ_s is obtained already for $\boldsymbol{e} = \boldsymbol{0}$. The presented numerical evidence (and other examples, not reported here) suggests that the approximation made by imposing the degree bounds (7) has small effect on the design even with $\boldsymbol{e} = 0$; we conjecture that for all practical purposes we can safely take $\boldsymbol{e} \leq (1, 1)$.

Example 2. This is a fan filter, with passband and the stopband defined by

$$\begin{aligned} \mathcal{D}_{p2} &= \{\omega_{1,2} \mid 2\cos\omega_1 - \cos\omega_2 - 1 \geq 0, \ \cos\omega_2 \geq 0\}, \\ \mathcal{D}_{s2} &= \{\omega_{1,2} \mid -2\cos\omega_1 + \cos\omega_2 \geq 0\} \\ & \quad \cup \{\omega_{1,2} \mid -\cos\omega_2 - 0.7 \geq 0\}. \end{aligned}$$

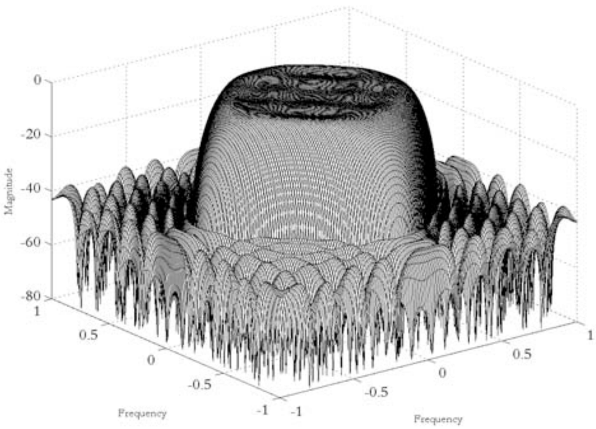

Fig. 3. Magnitude response of the filter from Example 1, with $\boldsymbol{e} = \boldsymbol{0}$.

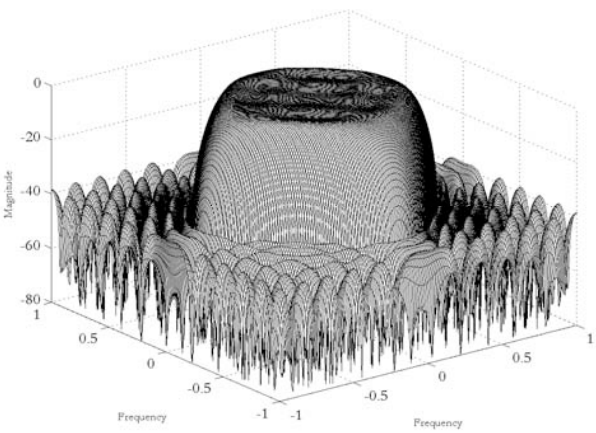

Fig. 4. Magnitude response of the filter from Example 1, with $\boldsymbol{e} = \boldsymbol{1}$.

Now, the passband is defined by the positivity of two polynomials; this will increase the complexity of the SDP problem since there are more parameter matrices in (8). Also, the stopband is a union of two domains and so we have to modify (16) by imposing the first two constraints for each domain in the union; this will also increase the complexity. Indeed, the design time is about 60 seconds, greater than for Example 1. The optimal stopband error is $\gamma_s = 0.020837$.

Comparison with linear programming method. Let $\mathcal{G} \in [-\pi, \pi]^2$ be a discrete set of frequency points, typically a grid. We approximate the design problem (16) by its discretized version, by imposing the positivity conditions not on compact frequency domains, but on their intersection with $\mathcal{G}$. For a given frequency point $\boldsymbol{\omega}$, each constraint is linear in the coefficients of $H^e(\boldsymbol{z})$. Thus, a linear programming (LP) problem is obtained. We build $\mathcal{G}$ as a union of two sets of points. The first is a regular grid, with step π/L, covering the half plane $[-\pi, \pi] \times [0, \pi]$. The second set consist of $2L$ points on each of the borders of $\mathcal{D}_p$ and $\mathcal{D}_s$; these points are necessary if we want to avoid the larger errors that typically appear near the borders.

We have solved the resulting LP program using SeDuMi, for several values of L. Some experimental results are given in Table 2. The first column shows the method. The second column shows the value γ_s as reported by the program. We compute the actual errors in the stopband and the passband on a fine grid with step $\pi/200$; the largest errors are reported in the third and fourth columns, respectively. For the LP method, the actual errors are larger than the ones imposed on the grid $\mathcal{G}$, a typical phenomenon. For our method, the actual errors are practically equal to the imposed ones, which suggests that the imposed bounds are tight. Moreover, for comparable execution times, our method designs filters with better performance than those obtained with the LP method.

Method	γ_s reported	γ_s on fine grid	γ_p on fine grid	execution time (sec)
LP, $L = 20$	0.011332	0.013389	0.055416	20
LP, $L = 30$	0.011431	0.012893	0.053954	70
LP, $L = 40$	0.012019	0.013043	0.050714	135
LP, $L = 50$	0.012057	0.012565	0.050732	200
SDP, $\boldsymbol{e} = \boldsymbol{0}$	0.012496	0.012489	0.0499995	40
SDP, $\boldsymbol{e} = \boldsymbol{1}$	0.012303	0.012301	0.0499990	85

Table 2. Comparison of the SDP and LP methods, for Example 1.

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