AUTOMATIC SMOOTHING OF PERIODOGRAMS

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ABSTRACT

Thresholding the cepstrum associated with the periodogram is a smoothing technique that appears to be very useful for variance reduction. Here the thresholding is performed via the methods SThresh and EbayesThresh. They both work fine in the broadband spectra case, even if some of the data is missing. The SThresh method appears to be more efficient as it shows a smaller variance and is faster computationally. The smoothing methods are also shown to perform well on a real-life broadband signal.

1. INTRODUCTION

Consider an observed sample $\{y_t\}_{t=0}^{N-1}$ of a stationary, discrete-time, real-valued signal with power spectral density, or spectrum, $\Phi(\omega)$ ($\omega \in (-\pi, \pi]$). For notational simplicity, let $\{\Phi_p\}$ denote the values of the spectrum at the Fourier frequency grid points:

$$\omega_p = \frac{2\pi}{N} p \qquad p = 0, ..., N-1. \tag{1}$$

The periodogram estimate of Φ_p is given by (see, e.g., [1] [2]):

$$\hat{\Phi}_p = \frac{1}{N} \left| \sum_{t=0}^{N-1} y(t) e^{-i\omega_p t} \right|^2 \qquad p = 0, ..., N-1. \tag{2}$$

The high variance of the periodogram, which does not converge to zero as N increases, but to Φ_p^2, has resulted in many forms of smoothing techniques to overcome this problem. These techniques mostly demand that the selection of the window and its span is performed with prudence and for this there are no clear guidelines.

Two recent papers, [3] and [4], have proposed the use of cepstrum as point of departure for a new smoothing technique for nonparametric spectral estimators. Cepstrum has been used before in spectral estimation, but in parametric estimation of ARMA-parameters (see, e.g., [5] [6] [7]).

The cepstrum was first introduced in [8] and is used not only in spectral estimation, but also in a wide variety of applications: speech processing, filter design, image processing, geology and much more. See [9] [10] for rather extensive lists of papers on cepstrum and its applications.

The cepstrum of a signal $\{y(t)\}$ can be defined as

$$c_k = \frac{1}{N} \sum_{p=0}^{N-1} \ln(\Phi_p) e^{i\omega_k p} \quad k = 0, ..., N-1 \tag{3}$$

where it is assumed that $\Phi_p > 0, \ \forall p$. Following [3] [4], the sequence $\{c_k\}$ will also be called *vocariance* sequence, due to its resemblance with the covariance sequence.

Thresholding the vocariance sequence and taking exponential function of the inverse Fourier transform results in a smoothed spectrum. The thresholding can be performed in many ways, two of which are Simple Thresholding, abbreviated SThresh, ([3] [4]) and Empirical Bayes Thresholding (EbayesThresh) ([11] [12]). These methods are used in this article to perform smoothing in a real-life data example, in a Monte Carlo simulation with full data, and a Monte Carlo simulation where ten percent of the data is missing.

2. METHODS

2.1. Simple Thresholding

The vocariance sequence has several interesting features, one of which is its mirror symmetry:

$$c_{N-k} = c_k \qquad k = 0, 1, ..., \frac{N}{2}, \tag{4}$$

which means that only half of the sequence, $c_0, ..., c_{\frac{N}{2}}$, is distinct. The other half is obtained from $c_1, ..., c_{\frac{N}{2}-1}$ via (4).

Using the periodogram estimate in (2), a common estimate of the vocariance is given by [3] [4]:

$$\hat{c}_k = \frac{1}{N} \sum_{p=0}^{N-1} \ln(\hat{\Phi}_p) e^{i\omega_k p} + \gamma \delta_{k,0} \qquad k = 0, ..., M \tag{5}$$

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where

$$\delta_{k,0} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{else,} \end{cases} \tag{6}$$

$M = \frac{N}{2}$ and $\gamma = 0.577216$ (Euler's constant).

It can be shown (see, e.g., [3]) that for large samples, the estimated vocariances $\{\hat{c}_k\}_{k=0}^{M}$ are independent normally distributed random variables:

$$\hat{c}_k \sim \mathcal{N}(c_k, s_k^2) \tag{7}$$

with

$$s_k^2 = \begin{cases} \frac{\pi^2}{3N} & \text{if } k = 0, M \\ \frac{\pi^2}{6N} & \text{if } k = 1, .., M-1. \end{cases} \tag{8}$$

With the above equations in mind, the idea behind cepstrum thresholding is straightforward. Let $\tilde{c}_k$ be a new estimate of c_k and note that $\tilde{c}_k = 0$ has a mean squared error (mse) equal to c_k^2. This estimate is preferred to $\hat{c}_k$ as long as $c_k^2 \leq s_k^2$. Now let

$$S = \{k \in [0, M] \mid c_k^2 \leq s_k^2\} \tag{9}$$

and let $\tilde{S}$ be an estimate of the set S. Thresholding $\{\hat{c}_k\}_{k \in \tilde{S}}$ gives new estimates of c_k:

$$\tilde{c}_k = \begin{cases} 0 & \text{if } k \in \tilde{S} \\ \hat{c}_k & \text{else} \end{cases} \qquad k = 0, ...M. \tag{10}$$

A good estimate of S is given by (see [4] for details):

$$\tilde{S} = \{k \in [0, M] \mid |\hat{c}_k| \leq \mu s_k\} \tag{11}$$

where the parameter μ controls the risk of concluding that $|c_k|$ is "significant" when this is not true, the so called "false alarm probability". The following values of μ are recommended in [4] for $N \in (128, 2048)$:

$$\mu = \mu_0 + \frac{N - 128}{1920} \tag{12}$$

where

$$\mu_0 = \begin{cases} 4 & \text{for a broadband signal with small dynamic range} \\ 3 & \text{for a broadband signal with large dynamic range} \\ 2 & \text{for a narrowband signal with vary large dynamic range.} \end{cases} \tag{13}$$

This means that μ will belong to the interval (μ_0, μ_0+1). For other intervals of the sample length, N, similar rules can be given.

Since μ has to be chosen manually, the method is not fully automatic. However, the a priori information needed for the selection of μ is modest; a quick inspection of the periodogram tells whether the signal is broad- or narrowbanded and whether its dynamic range is large or small. The SThresh scheme can thus be considered to be essentially automatic. A suggestion for a fully automatic scheme is given in [4].

Finally, the smoothed spectral estimate corresponding to $\{\tilde{c}_k\}$ is given by:

$$\tilde{\Phi}_p = \exp\left[\sum_{k=0}^{N-1} \tilde{c}_k e^{-i\omega_p k}\right] \qquad p = 0, ..., N-1. \tag{14}$$

2.2. Empirical Bayes Thresholding

EbayesThresh is a fully automatic method aimed to estimate sparse sequences observed in Gaussian white noise. Simulations have shown that the method has an excellent performance (see [11]) and there are implementations of the method in both the R language ([13]) and MATLAB™([14]).

Very briefly, EbayesThresh uses—as the name suggests—an empirical Bayes approach to solve the estimation problem. Under a Bayesian model, each parameter c_k is assumed to be zero with probability $(1 - w)$, while, with probability w, c_k is considered to be drawn from a symmetric heavy-tailed density. w is chosen automatically from the data using a marginal maximum likelihood approach and then inserted into the Bayesian model. The (soft) threshold is thus given by the choice of w. For a detailed explanation of the EbayesThresh method, see, e.g., [11] and [12].

In this article, EbayesThresh was performed on the vocariance sequence $\hat{c}_k$, giving a thresholded sequence $\tilde{c}_k^{ebayes}$. The smoothed spectral estimate was then given by (14) similarly to the SThresh method.

3. EXAMPLES

The two methods SThresh and EbayesThresh were applied to three types of signals:

- A broadband moving average (MA) signal with small dynamic range with MA equation, below called the full data case:

$$y(t) = e(t) + 0.55e(t-1) + 0.15e(t-2) \qquad t = 0, ..., N-1. \tag{15}$$

- The broadband MA in (15), but with ten percent of the data missing, i.e. set randomly to zero, below called the missing data case.
- A real-life data set, a sampled ocean wave, taken from [2].

The EbayesThresh method was called as:

```
c_ebayes = ebayesthresh(data=c_hat, prior=
'laplace', a=[], bayesfac=0, sdev=[],
verbose=0, threshrule='median', isotone=0).
```

This means that `c_hat` ($= \hat{c}_k$) was thresholded using a Laplace prior, which scale factor (parameter `a`) was estimated using marginal maximum likelihood. The Bayes factor threshold was not used, the standard deviation was estimated using median absolute deviation from zero, no verbose mode was used, the median thresholding rule was applied, and, with `isotone=0`, the monote marginal likelihood estimation was not used. See, e.g., [12] [13] [14] for more details on this method. The parameters were set accordingly to the recommendations in [11] and mostly using their default values. The SThresh method was used with $\mu_0 = 4$.

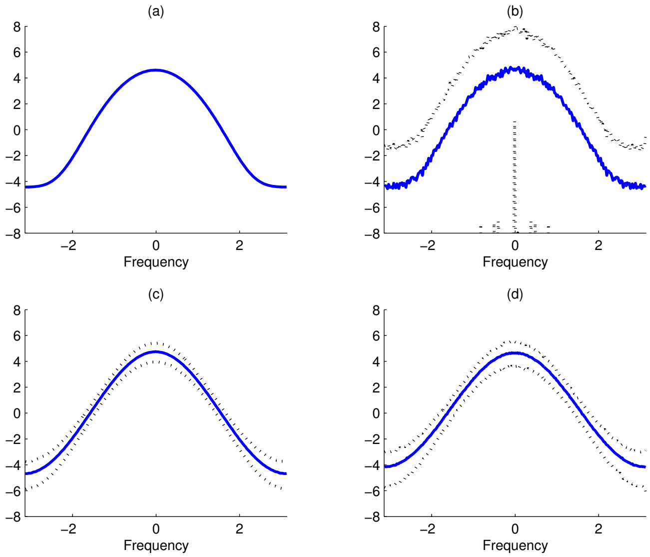


Fig. 1. Log-spectrum of the MA signal: a) the true spectrum, b) the averaged periodogram with its standard deviations, and c) and d) the averaged smoothed periodograms and their standard deviations, obtained via SThresh and EbayesThresh.

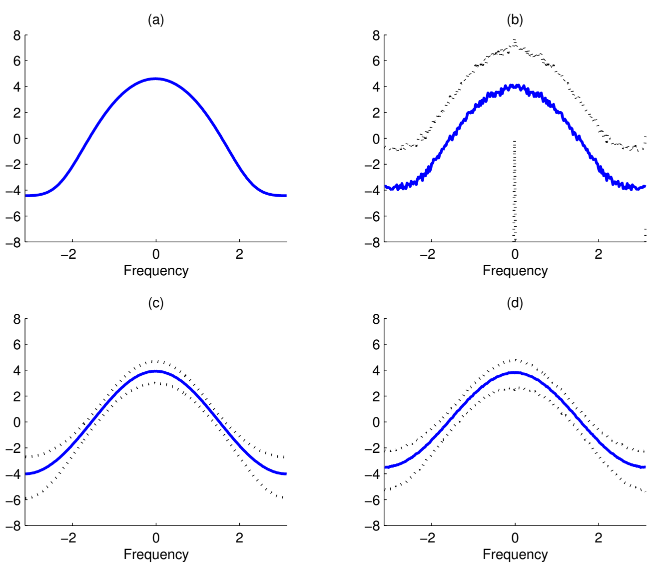
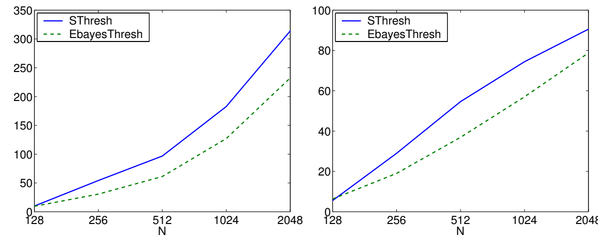


Fig. 2. Log-spectrum of the MA signal when ten percent of the data is missing: a) the true spectrum, b) the averaged periodogram with its standard deviations, and c) and d) the averaged smoothed periodogram and their standard deviations, obtained via SThresh and EbayesThresh.

3.1. The Simulated Examples

The simulations, both for the full data and the missing data case, were performed as 1000 Monte Carlo runs for $N = 128, 256, 512, 1024$ and 2048. The periodograms were averaged and the averaged periodogram together with its standard



(a) Full data case. (b) Missing data case.

Fig. 3. The ratio $\text{TV}(\hat{\mathbf{c}})/\text{TV}(\tilde{\mathbf{c}})$, versus N, for the two thresholding methods in the full data and the missing data case.

deviations can be seen in Fig. 1(b) for the full data and in Fig. 2(b) for the missing data case. Fig. 1(a) and 2(a) displays the true spectrum of the MA signal. Fig. 1 also contains the averaged smoothed periodograms and their standard deviations. In Fig. 2 these plots can also be seen for the missing data case. The plots show the same tendencies: the standard deviations are significantly decreased for the smoothed periodograms compared to the non-smoothed periodogram and the variance for the SThresh method is slightly smaller than for the EbayesThresh method.

Fig. 3(a) and Fig. 3(b) show the ratios of the total variances of the periodogram and the thresholded periodogram, $\text{TV}(\hat{\mathbf{c}})/\text{TV}(\tilde{\mathbf{c}})$, for the two cases of simulated signals. The TV is defined as:

$$\begin{aligned} \text{TV}(\hat{\mathbf{c}}) &\triangleq \sum_{k=0}^{N-1} E(\hat{c}_k - c_k)^2 \\ &= \sum_{k=0}^{N-1} E\left[\ln(\hat{\Phi}_p) - \ln(\Phi_p)\right]^2 \end{aligned} \tag{16}$$

and similarly for $\text{TV}(\tilde{\mathbf{c}})$. A larger ratio is thus better. See e.g. [4] for a more detailed discussion on the TV.

In both the full and the missing data case, SThresh shows a better performance than EbayesThresh, even though the difference is not very large for smaller values of N. SThresh is also generally significantly faster, in the full data simulation around 170 times faster.

3.2. The Ocean Wave Data Example

The real-life data example is a time series recorded in the Pacific Ocean by a wave-follower. Every 1/4 second the sea level is measured as the wave-follower moves up and down following the water surface. The data consists of $N = 1024$ data points and were low-pass filtered using an antialiasing filter with a cutoff frequency of approximately 1 Hz. The data was originally collected to investigate whether the rate at which the spectrum decreases in the interval 0.2 to 1.0 Hz was consistent with a physical model. The behaviour of frequencies above 1 Hz was of little interest since it was mainly determined by instrumentation and preprocessing. For more in-

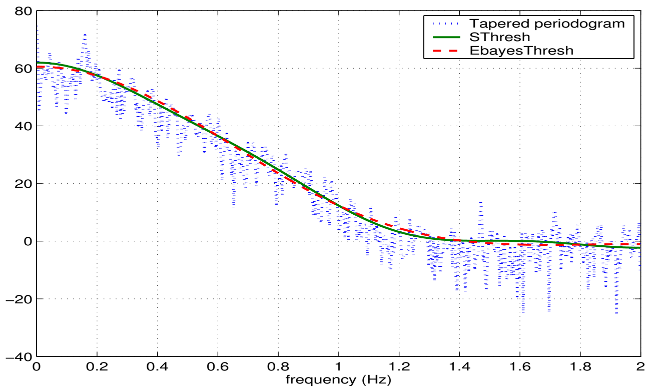


Fig. 4. Log-spectrum of the tapered periodogram of the ocean wave data together with the smoothed periodograms obtained via SThresh and EbayesThresh.

formation on the ocean wave data, see [2]. The data were preprocessed as in [2], i.e. using a discrete prolate spheroidal sequence data taper to prevent leakage, before being smoothed. The results can be seen in Fig. 4. The tapered periodogram is highly erratic whereas both the smoothed curves show little variance. Both SThresh and EbayesThresh give smoothed periodograms where the slope in the range 0.2 to 1.0 Hz can be easily computed, which was the purpose of this exercise.

4. CONCLUSIONS

Apparently both SThresh and EbayesThresh perform well as smoothing methods: the variance of the raw periodogram is significantly decreased. SThresh appears to perform better as its ratio $\mathrm{TV}(\hat{\mathbf{c}})/\mathrm{TV}(\tilde{\mathbf{c}})$ is superior to that of EbayesThresh, both for the full data case and the missing data case. SThresh also has the following additional advantages:

- It is very fast, around 170 times faster than EbayesThresh.
- It is more intuitive to explain and is easier to implement than EbayesThresh.
- It has fewer user parameters than EbayesThresh: only μ has to be selected, whereas for EbayesThresh there are several parameters that can be tuned.

Presumably EbayesThresh might perform better, if its parameters are tuned. But since the method was supposed to be automatic, little effort was spent on tuning.

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