Estimation of Instantaneous Frequency and Instantaneous Bandwidth via Adaptive Signal Decomposition

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Abstract: In this work, we derive general formulae for the instantaneous frequency (IF) and instantaneous bandwidth (IB) according to matching pursuit (MP) signal decomposition, irrespective of the kind of dictionary used. We show that via MP decomposition, the IF is now exactly the weighted average IF of the decomposed signal and that it is also always real-valued. In addition, the IB is always positive for all dictionaries with Gaussian envelopes and arbitrary polynomial phase.

1. INTRODUCTION

Consider the complex signal $z(t)$, where both $a(t)$ and $\phi(t)$ are real:

$$z(t) = \sum_{n=1}^{N} a_n(t) e^{j\phi_n(t)} = a(t)e^{j\phi(t)} . \tag{1}$$

The instantaneous frequency (IF) is defined as the derivative of the phase ($\phi'(t)$) and the instantaneous bandwidth (IB) is given by the absolute value of the log-amplitude [1] - i.e., $|a'(t)/a(t)|$. In time-frequency analysis, these are interpreted as low-order conditional spectral moments of the time-frequency distribution of a signal [2]. So given a joint time-frequency distribution, $P_z(t,\omega)$, the conditional spectral moments are

$$<\omega^l>_t = \frac{1}{P_z(t)} \int_{-\infty}^{+\infty} \omega^l P_z(t,\omega) d\omega \tag{2}$$

$$\text{for } P_z(t) = \int_{-\infty}^{+\infty} P_z(t,\omega) d\omega. \tag{3}$$

Then the IF and IB are respectively $<\omega>_t$ and $\sigma_{\omega|t} = \sqrt{(<\omega^2>_t - <\omega>_t^2)}$. Recently, Loughlin and Davidson [1] suggested the use of the weighted average IF (WAIF) and the modified IB. Investigations of these two quantities revealed that the WAIF does not exceed the spectral support of the signal $z(t)$ and also that the modified IB is always positive. These properties are useful for physical interpretation.

So in this paper we begin with a brief review of MP signal decomposition theory in section 2. In section 3 we first obtain expressions for the IF and IB for a multi-component signal, and then present similar expressions for the WAIF and the modified IB. In section 4, under MP signal decomposition, we derive expressions for the first conditional spectral moment $\left(<\omega>_t\right)$ and the conditional spectral variance $\left(\sigma^2_{\omega|t}\right)$ - irrespective of the chosen dictionary. In section 5, two different signals will be analyzed and the above results will be verified in practice. Finally, in section 6 we will present some conclusions.

2. REVIEW MP DECOMPOSITION THEORY

Mallat and Zhang [3] have proposed an adaptive signal decomposition – matching pursuit (MP). This method is based on a dictionary that contains a family of functions called *elementary functions* or *time-frequency atoms*. The MP decomposition of the signal $z(t)$ after M iterations can be written as follows:

$$z(t) = \sum_{n=0}^{M-1} c_n g_{\gamma_n}(t) + R^M z(t) \tag{4}$$

where $R^M z(t)$ is the residue after M times signal decomposition; $g_{\gamma_n}(t)$ is the (unit-norm) time-frequency atom that belongs to the particular dictionary; the coefficient $c_n = \int_{-\infty}^{+\infty} R^n z(t) \cdot g^*_{\gamma_n}(t) dt$ is the inner product of the functions $\left(R^n z(t), g_{\gamma_n}(t)\right)$, where "*" represents the complex conjugate; and the parameter γ_n refers to the specific atom's parameter set. When the number of iterations is infinitive, then the residue will be zero, and so we can say

$$\lim_{M\to\infty} R^M z(t) = 0 \Rightarrow z(t) = \sum_{n=0}^{+\infty} c_n g_{\gamma_n}(t) . \qquad (5)$$

So Mallat and Zhang [3] defined the MP distribution

$$E_z(t,\omega) = \sum_{n=0}^{+\infty} |c_n|^2 W_{g_{\gamma_n}}(t,\omega) \qquad (6)$$

where $W_{g_{\gamma_n}}(t,\omega)$ is the Wigner-Ville Distribution (WVD) of the appropriate atom with

$$W_{g_{\gamma_n}}(t,\omega) = \int_{-\infty}^{+\infty} g_{\gamma_n}\left(t+\frac{\tau}{2}\right) g^*_{\gamma_n}\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau} d\tau . \qquad (7)$$

3. THE WAIF AND THE MODIFIED IB

Here we will now derive general expressions for the IF and IB of the *N*-component signal in (1):

$$\begin{aligned}\phi'(t) &= \frac{\sum_{n=1}^{N} a_n^2(t)\phi_n'(t)}{a^2(t)} \\ &+ \frac{\sum_{n=1}^{N-1}\sum_{m=n+1}^{N} a_n(t)a_m(t)(\phi_n'(t)+\phi_m'(t))\cos(\Delta\phi_{mn}(t))}{a^2(t)} \\ &+ \frac{\sum_{n=1}^{N-1}\sum_{m=n+1}^{N} (a_n(t)a_m'(t)-a_n'(t)a_m(t))\sin(\Delta\phi_{mn}(t))}{a^2(t)},\end{aligned} \qquad (8)$$

$$\begin{aligned}\frac{a'(t)}{a(t)} &= \frac{\sum_{n=1}^{N} a_n(t)a_n'(t)}{a^2(t)} \\ &+ \frac{\sum_{n=1}^{N-1}\sum_{m=n+1}^{N} (a_n'(t)a_m(t)+a_n(t)a_m'(t))\cos(\Delta\phi_{mn}(t))}{a^2(t)} \\ &- \frac{\sum_{n=1}^{N-1}\sum_{m=n+1}^{N} a_n(t)a_m(t)(\phi_m'(t)-\phi_n'(t))\sin(\Delta\phi_{mn}(t))}{a^2(t)}\end{aligned} \qquad (9)$$

where $a^2(t) = \sum_{n=1}^{N} a_n^2(t) + \sum_{n=1}^{N-1}\sum_{m=n+1}^{N} 2a_n(t)a_m(t)\cos(\Delta\phi_{mn}(t))$, and $\Delta\phi_{mn}(t) = \phi_m(t) - \phi_n(t)$. But the presence of the oscillatory terms in (8) and (9) causes difficulties in interpretation of the IF and the IB [4]. So Loughlin and Davidson [1] proposed the WAIF of the individual components for analysis of multi-component signals. Hence, for the *N*-component signal $z(t)$, the WAIF is

$$\overline{\omega}(t) = \frac{\sum_{n=1}^{N} a_n^2(t)\phi_n'(t)}{\sum_{n=1}^{N} a_n^2(t)} . \qquad (10)$$

They also suggested a new conditional spectral variance and so similarly modified the IB for a two component signal:

$$\sigma_{\omega|t}^2 = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)a_2^2(t)(\phi_1'(t)-\phi_2'(t))^2}{(a_1^2(t)+a_2^2(t))^2} . \qquad (11)$$

Note that this expression for $\sigma_{\omega|t}^2$ is never negative. Furthermore, as the spectral separation between the IF's increases or the derivative of amplitude modulations become different from zero, so the modified IB increases. This is what we would reasonably expect.

4. IF AND IB VIA MP DECOMPOSITION

Now suppose the signal $z(t)$ is decomposed via the MP algorithm. So it can be expressed as

$$z(t) = \sum_{n=0}^{+\infty} c_n g_{\gamma_n}(t) = \sum_{n=0}^{+\infty} a_n(t) e^{j\phi_n(t)} = \sum_{n=0}^{+\infty} z_n(t) \qquad (12)$$

where $z_n(t) = a_n(t)e^{j\phi_n(t)}$, and the parameters a_n and ϕ_n are evaluated according to the chosen time-frequency elementary function. So the MP distribution can be written as

$$E_z(t,\omega) = \sum_{n=0}^{+\infty} |c_n|^2 W_{g_{\gamma_n}}(t,\omega) = \sum_{n=0}^{+\infty} W_{z_n}(t,\omega) . \qquad (13)$$

Now from (2) and (3) the first conditional spectral moment is

$$<\omega>_t = \frac{\int_{-\infty}^{+\infty} \omega E_z(t,\omega)d\omega}{\int_{-\infty}^{+\infty} E_z(t,\omega)d\omega} = \frac{\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty} \omega W_{z_n}(t,\omega)d\omega}{\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty} W_{z_n}(t,\omega)d\omega} \qquad (14)$$

where

$$\begin{aligned}&\int_{-\infty}^{+\infty} \omega W_{z_n}(t,\omega)d\omega \\ &= \int_{-\infty}^{+\infty} a_n(t+\frac{\tau}{2})a_n(t-\frac{\tau}{2})e^{j(\phi_n(t+\frac{\tau}{2})-\phi_n(t-\frac{\tau}{2}))}(j2\pi)\delta'(\tau)d\tau \\ &= (2\pi)a_n^2(t)\phi_n'(t)\end{aligned} \qquad (15)$$

and

$$\int_{-\infty}^{+\infty} W_{z_n}(t,\omega)\,d\omega$$

$$= \int_{-\infty}^{+\infty} a_n(t+\frac{\tau}{2})a_n(t-\frac{\tau}{2})e^{j(\phi_n(t+\frac{\tau}{2})-\phi_n(t-\frac{\tau}{2}))}(2\pi)\delta(\tau)d\tau$$

$$= (2\pi)a_n^2(t) \tag{16}$$

with $\delta(\tau)$ and $\delta'(\tau)$ the Dirac-delta function and its derivative, respectively. So the first conditional spectral moment of the MP distribution in (14) now takes the following form:

$$<\omega>_t = \frac{\sum_{n=0}^{+\infty} a_n^2(t)\phi_n'(t)}{\sum_{n=0}^{+\infty} a_n^2(t)}. \tag{17}$$

Although it has been shown that the IF only equals the WAIF under certain strict conditions [5]-[6], the result above is identical to the WAIF (in (10)) for the MP decomposed signal - i.e. $<\omega>_t = \overline{\omega}(t)$. Because the convergence of the MP signal decomposition is not dependent upon the type of time-frequency atom used (although the actual speed of convergence is), so (17) is valid for any atom. In addition, (17) ensures that the IF is always real valued.

Now consider the second-order moment and then the IB:

$$<\omega^2>_t = \frac{\int_{-\infty}^{+\infty}\omega^2 E_z(t,\omega)d\omega}{\int_{-\infty}^{+\infty}E_z(t,\omega)d\omega} = \frac{\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty}\omega^2 W_{z_n}(t,\omega)d\omega}{\sum_{n=0}^{+\infty}\int_{-\infty}^{+\infty}W_{z_n}(t,\omega)d\omega} \tag{18}$$

where

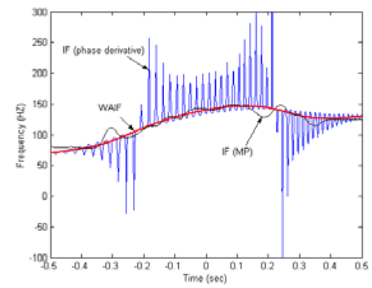
$$\int_{-\infty}^{+\infty}\omega^2 W_{z_n}(t,\omega)d\omega$$

$$= \int_{-\infty}^{+\infty} a_n(t+\frac{\tau}{2})a_n(t-\frac{\tau}{2})e^{j(\phi_n(t+\frac{\tau}{2})-\phi_n(t-\frac{\tau}{2}))}(-2\pi)\delta''(\tau)d\tau$$

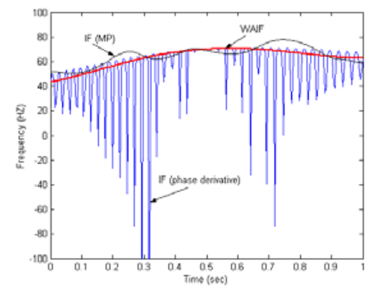
$$= (2\pi)\left(\frac{1}{2}\left(a_n'^2(t) - a_n''(t)a_n(t)\right) + a_n^2(t)\phi_n'^2(t)\right) \tag{19}$$

and $\delta''(\tau)$ is the second derivative of the Dirac-delta function. Thus the second-order moment is

$$<\omega^2>_t = \frac{\sum_{n=0}^{+\infty}\frac{1}{2}\left(a_n'^2(t)-a_n''(t)a_n(t)\right)}{\sum_{n=0}^{+\infty}a_n^2(t)} + \frac{\sum_{n=0}^{+\infty}a_n^2(t)\phi_n'^2(t)}{\sum_{n=0}^{+\infty}a_n^2(t)}. \tag{20}$$



(a) IF estimation for the signal in (23).



(b) IF estimation for the signal in (24).

Fig. 1. IF estimation for the two different signals in (23) and (24) (see (a) and (b)), calculated by the three different methods: (i) phase derivative $\phi'(t)$, via (8) (the blue plot); WAIF $\overline{\omega}(t)$, via (10) (the red plot); and $<\omega>_t$ via adaptive (MP, M = 16 iterations) signal decomposition (the black plot).

So from (17) and (20) we now have a general formula for the conditional spectral variance:

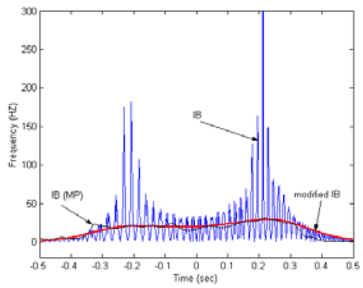
$$\sigma_{\omega|t}^2 = (<\omega^2>_t - <\omega>_t^2)$$

$$= \frac{\sum_{n=0}^{+\infty}\frac{1}{2}(a_n'^2(t)-a_n''(t)a_n(t))}{\sum_{n=0}^{+\infty}a_n^2(t)} + \frac{\sum_{n=0}^{+\infty}\sum_{m=n+1}^{+\infty}a_n^2(t)a_m^2(t)(\phi_n'(t)-\phi_m'(t))^2}{(\sum_{n=0}^{+\infty}a_n^2(t))^2}. \tag{21}$$

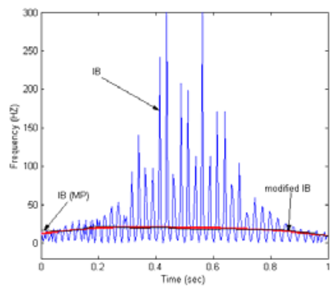
For many bilinear distributions belonging to the Cohen class (such as the WVD) there is no guarantee that the conditional spectral variance will always be positive. But for any MP elementary function with a Gaussian shaped envelope the IB is never negative. The general elementary function with a Gaussian envelope is

$$g_{\gamma_n}(t) = \frac{1}{\sqrt{s_n}}g\left(\frac{t-u_n}{s_n}\right)e^{j\phi_n(t)} \tag{22}$$

where $g(t) = 2^{1/4}e^{-\pi t^2}$ and $\gamma_n = (s_n, u_n, \phi_n)$ is the atom parameter set. The parameter $\phi_n(t)$ may be a polynomial of order L – i.e., $\phi_n(t) = \sum_{l=0}^{L}\beta_{n,l}t^l$. For $L=1$ this elementary function is just a Gaussian atom and for $L=2$ it is a chirplet atom.



(a) IB estimation for the signal in (23).



(b) IB estimation for the signal in (24).

Fig. 2. IB estimation for the two different signals in (23) and (24) (see (a) and (b)), calculated by the three different methods: (i) via the absolute value of the log-amplitude in (9) (the blue plot); via the modified IB in (11) (the red plot); and via $\sigma_{\omega|t}$ based on adaptive (MP, M = 16 iterations) signal decomposition (the black plot).

5. SIMULATIONS

In this section, we give examples to test the new eqns derived in section 4. We will compare these with both the original IF and IB ((8) and (9)) and the formulae suggested by Loughlin and Davidson ((10) and (11)). The advantage of using simulated signals is that we already know the amplitude and phase analytically and so we can easily plot the aforementioned formulae and compare results. Now consider the following two (2-component) signals:

$$z_1(t) = e^{-15t^2} e^{j2\pi(50t^2+150t)} + \frac{1}{2} e^{j2\pi(30t^2+100t)} \tag{23}$$

$$\begin{aligned} z_2(t) &= e^{-5(t-0.5)^2} e^{j20\pi((t-0.5)^2+5t)} \\ &\quad + e^{-10(t-0.5)^2} e^{j10\pi((t-0.5)^2+18t)}. \end{aligned} \tag{24}$$

The signal (23) differs from (24) insofar as both components in (24) have time-varying amplitudes, while this is true for only the first component in (23). These signals were then decomposed according to the MP algorithm (using the Gaussian elementary function) in order to obtain $\{a_n(t)\}$ and $\{\phi_n(t)\}$ in (12). Three different estimates of the IF and IB are now calculated and plotted in Figs 1 and 2 – first, via the usual definitions of $\phi'(t)$ in (8) and $\left|\frac{a'(t)}{a(t)}\right|$ in (9); second, via the definitions of the WAIF $\left(\bar{\omega}(t)\right)$ in (10) and the modified conditional spectral variance $\left(\sigma^2_{\omega|t}\right)$ in (11); and third, via the practical estimation of $<\omega>_t$ in (17) and $\sigma^2_{\omega|t}$ in (21), but now based on MP.

As the two signals in (23) and (24) are analytically known, we can easily calculate and plot the first two theoretical estimates above for comparison. It is clear to see that what we have computed as the first conditional spectral moment based on the MP distribution has a very good agreement with the true WAIF. In addition, there is also good agreement between the modified IB and the IB evaluated via MP.

6. DISCUSSION AND CONCLUSIONS

It is often claimed that instantaneous frequency, taken as the derivative of the phase of the signal, is appropriate or meaningful only for mono-component signals, and that for multi-component signals a weighted average of individual instantaneous frequencies should be used – i.e., the WAIF. In this paper, we have shown that the first conditional spectral moment, as computed via MP, is exactly the WAIF. Two different synthetic signals have been analysed and we can see from the simulations that what we compute as the first conditional spectral moment approaches (as the iterations increase) the true WAIF, as the theory predicts. Although in many cases the second conditional spectral moment is not positive (and so also the conditional spectral variance), and this makes the usual interpretation of these quantities impossible. But in this paper we have proved that with MP decomposition based on any elementary function with a Gaussian envelope, both these parameters are guaranteed to always be positive, and so also the IB estimate. Finally, the issue of what dictionary minimises the IF and IB estimation error is still an open question.

7. REFERENCES

[1] P. J. Loughlin, and K. L. Davidson, "Modified Cohen-Lee time-frequency distributions and instantaneous bandwidth of multicomponent signals," *IEEE Trans. Signal Processing*, vol. 49, no. 6, pp. 1153 -1165, June 2001.

[2] L. Cohen, "Time-frequency distributions- a review," *Proceedings of the IEEE*, vol. 77, no. 7, pp. 941 -980, July 1989.

[3] S.G. Mallat, and Z. Zhang, "Matching pursuit with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397 -3415, Dec. 1993

[4] P. J. Loughlin, "Comments on the interpretation of instantaneous frequency," *IEEE Signal Processing Letters*, vol. 4, no. 5, pp. 123 -125, April 1997.

[5] W. Nho, and P. J. Loughlin, "When is instantaneous frequency the average frequency at each time?," *IEEE Signal Processing Letters*, vol. 6, no. 4, pp. 78 -80, April 1999.

[6] P. J. Loughlin, "The time-dependent weighted average instantaneous frequency," *IEEE International Symposium on Time-Frequency and Time-Scale Analysis*, pp. 97-100, Oct. 1998.