MULTIPLE WINDOW DECOMPOSITION OF TIME-FREQUENCY KERNELS USING A PENALTY FUNCTION FOR SUPPRESSED SIDELOBES

Maria Hansson

Mathematical Statistics, Centre for Mathematical Sciences
Box 118, SE-221 00 Lund, Sweden
phone: +46 46 222 49 53, email: mh@maths.lth.se

ABSTRACT

This paper presents multiple windows with suppressed sidelobes that approximate a time-frequency kernel. The multiple windows are given from the eigenvalue decomposition of the time-lag kernel. By using a penalty matrix both in time- and frequency domain and solving a generalized eigenvalue problem, the multiple windows are achieved. Examples are given and the resulting sidelobe suppression and the number of windows needed are analysed.

1. INTRODUCTION

The area of time-frequency analysis is well covered in the signal processing literature and a large number of time-frequency kernels are proposed for various types of applications. Using eigenvalues and eigenvectors of the rotated time-lag kernel the resulting multiple window spectrogram is the smoothed Wigner-Ville estimate, [1, 2]. To make this interpretation useful from the computational aspect the number of calculated spectrograms have to be reasonable, i.e., the number of eigenvalues differing from zero has to be few, [3, 4].

The phrase multiple windows were originally introduced by Thomson, [5], for the case of stationary processes with smooth spectra. The advantage of the Thomson multiple windows are the sidelobe suppression outside a predetermined frequency interval which gives the resolution. Other methods have been proposed, [6, 7]. In [7] a penalty function was introduced to suppress sidelobes of the spectrum outside a certain predetermined frequency interval when the shape of the spectrum had a specific shape.

In this paper we use the same idea to suppress the sidelobes of the multiple window spectrogram outside a certain frequency interval as well as a certain time interval. The idea can be used to suppress sidelobes of other time-frequency kernels and at the same time reduce the number of spectrograms needed to be calculated in a multiple window interpretation.

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2. SPECTROGRAM DECOMPOSITION OF TIME-FREQUENCY KERNELS

The time-frequency distribution of the signal $x(t)$ is calculated by

$$W(t,\omega) = \frac{1}{(2\pi)^2} \int\int G(t-t',\tau)G_x(t',\tau)dt'e^{-j\omega\tau}d\tau \quad (1)$$

where $G(t,\tau)$ is the time-lag kernel given from the inverse Fourier transform of the kernel $\phi(\theta,\tau)$ in the first variable,

$$G(t,\tau) = \int \phi(\theta,\tau)e^{-j\theta t}d\theta, \quad (2)$$

and $G_x(t,\tau)$ is the instantaneous autocorrelation

$$G_x(t,\tau) = x(t+(\tau/2))x^*(t-(\tau/2)). \quad (3)$$

The time-lag kernel is rotated according to

$$R(t_1,t_2) = G(\frac{-t_1-t_2}{2}, t_1-t_2), \quad (4)$$

and Eq. (1) is rewritten as an inner product, $W(t,\omega) =$

$$\int\int x(t+t_1)e^{-j\omega(t+t_1)}R(t_1,t_2)x^*(t-t_2)e^{j\omega(t+t_2)}dt_1dt_2. \quad (5)$$

The kernel $R(t_1,t_2)$ corresponds to a linear operator $\tilde{R}$ with a spectral representation

$$\tilde{R} = \sum_{i=1}^{I} \lambda_i P_i, \quad (6)$$

where λ_i are the corresponding eigenvalues and

$$\{P_i x = < x, u_i > u_i\}_{i=1}^{I} \quad (7)$$

are the orthonormal projections onto $\{u_i\}_{i=1}^{I}$ which are the corresponding eigenvectors. Using the eigenvalues and eigenvectors Eq. (5) is rewritten as a weighted sum of spectrograms,

$$W(t,\omega) = \sum_{i=1}^{I} \lambda_i |\int x(t+t_1)e^{-j\omega(t+t_1)}u_i^*(t_1)dt_1|^2. \quad (8)$$

Using discrete-time data, the solution is given from

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{q}_i, \quad i = 1 \ldots N, \tag{9}$$

where $\mathbf{R}$ is the sampled matrix of size $N \times N$ corresponding to $R(t_1, t_2)$. The eigenvalues are ordered according to $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. The eigenvectors corresponding to the I largest eigenvalues are used as windows, $\mathbf{u}_i = \mathbf{q}_i$, $i = 1 \ldots I$.

2.1. Example

A time-lag kernel defined as

$$G(t, \tau) = \frac{\omega_0}{2\pi}\text{sinc}(\frac{\omega_0 \tau}{2\pi}), \quad -\frac{t_0}{2} \leq t \leq \frac{t_0}{2}, \tag{10}$$

which corresponds to a time-frequency kernel (TF kernel) shaped as a box of level one both in the time- and frequency domain limited by $\frac{-\omega_0}{2} \leq \omega \leq \frac{\omega_0}{2}$ and $-\frac{t_0}{2} \leq t \leq \frac{t_0}{2}$. The sample size of the corresponding rotated time-lag matrix $\mathbf{R}$ is $N = 128$ and the box is limited by $\omega_0 = 2\pi 8/256 = 0.22$ and $t_0 = 8$. These values are used throughout the paper. The eigenvalues of the rotated kernel are seen in Fig. 1a. The number of eigenvalues that need to be included in the sum of spectrograms is $I = 86$ according to an simple criteria where I is the smallest number of windows fulfilling,

$$\frac{\sum_{i=1}^{I} \lambda_i}{\sum_{i=1}^{N} \lambda_i} \geq 0.99. \tag{11}$$

In Fig. 1b, the three first eigenvectors are depicted and in Fig. 1c and d, the resulting TF-kernel using $I = 86$ windows is depicted from the frequency axis and time axis respectively. The positive values of the TF-kernel are plotted in dB-scale. Negative values of the TF-kernel are not considered in the plots as the main aspect is the sidelobe height. The sidelobes of the kernel are around -20 dB. The aim is to decrease these outside a predetermined time-frequency area.

3. SIDELOBE SUPPRESSION USING PENALTY FUNCTIONS

3.1. Frequency sidelobe suppression

A penalty function defined as

$$S_L(\omega) = \begin{cases} L & |\omega| > \frac{\omega_K}{2} \\ 1 & |\omega| \leq \frac{\omega_K}{2} \end{cases} \tag{12}$$

is used to decrease the leakage from the sidelobes outside the frequency interval of width ω_K of the multiple windows in [7]. The corresponding covariance function is given by a Toeplitz covariance matrix, (time-independent), $\mathbf{R}_L$.

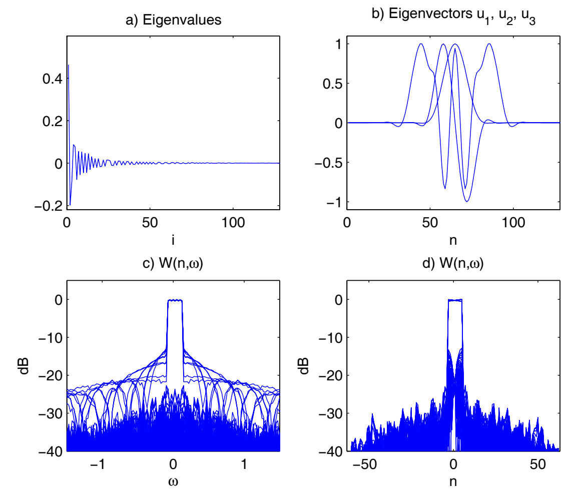

Fig. 1. a) Eigenvalues, b) The first three eigenvectors, c) The resulting TF kernel from the frequency axis, d) The resulting kernel from the time axis

The solution with respect to $\mathbf{u}_i$ is the set of eigenvectors of the **generalized** eigenvalue problem

$$\mathbf{R}\mathbf{q}_i = \lambda_i \mathbf{R}_L \mathbf{q}_i, \quad i = 1 \ldots N. \tag{13}$$

The ideal window functions fulfill the relationship

$$\sum_{i=1}^{I} \lambda_i \mid U_i(\omega) \mid^2 = S_L^{-1}(\omega), \quad -\pi \leq \omega \leq \pi, \tag{14}$$

and, if L is large, the sidelobes in the frequency domain of $|U_i(\omega)|^2$ outside the interval ω_K is suppressed by this factor.

The frequency domain suppression is exemplified by using the previous matrix $\mathbf{R}$ and a predetermined frequency interval of $\omega_K = 2\pi 60/256 = 1.47$ and the suppression parameter $L = 10^{\frac{30}{10}}$ (30 dB). The resulting eigenvalues and eigenvectors are depicted in Fig. 2a and b. The number of eigenvalues used to fulfill the criteria of Eq. (11) is $I = 23$. The sidelobe supression is clearly seen in Fig. 2c but in Fig. 2d the time domain sidelobes are now larger.

3.2. Time domain sidelobe suppression

To design a penalty function for the time domain we study the rotated time-lag kernel which has the structure of a time-dependent covariance matrix. A time penalty matrix is defined as in Fig. 3 where the aim is to suppress everything outside a certain time interval both in the time- and lag-variable. The level inside the square is one and outside the level is L. The size of the square is $t_K \times t_K$. An example is shown in Fig. 4 where the penalty matrix $\mathbf{R}_L$ of Eq. (14) is replaced with the matrix of Fig. 3 using $t_K = 60$ and $L = 10^{\frac{30}{10}}$ (30

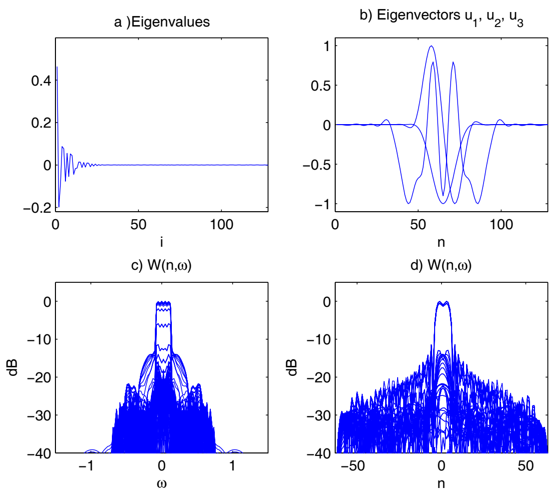

Fig. 2. a) Eigenvalues, b) The first three eigenvectors, c) The resulting TF kernel from the frequency axis, d) The resulting kernel from the time axis

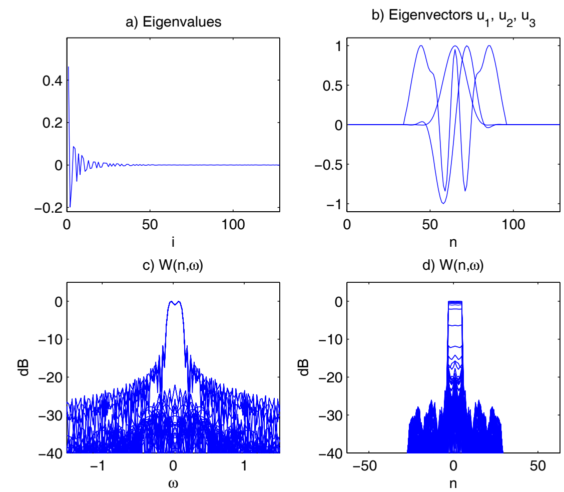

Fig. 4. a) Eigenvalues, b) The first three eigenvectors, c) The resulting TF kernel from the frequency axis, d) The resulting kernel from the time axis

dB). The resulting used number of windows is $I = 43$ and it is seen that the suppression in the time domain of the TF kernel (Fig. 4d) is large but in the frequency domain (Fig. 4c) no suppression is received. The conclusion of these two exemples is that the two penalty functions should be combined.

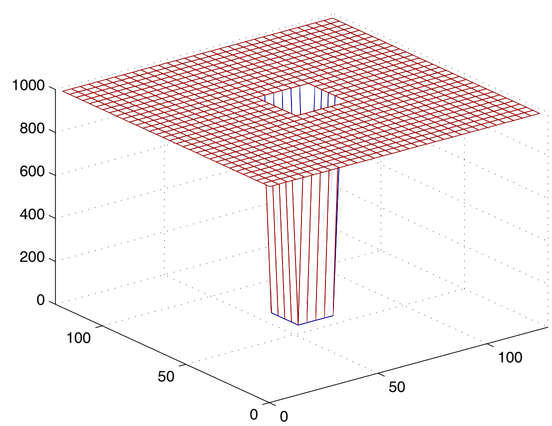

Fig. 3. Time domain penalty function

3.3. Time- and frequency sidelobe suppression

The previous examples show clearly that a penalty function suppressing both in time and frequency is needed. One suggestion is to combine the two penalty matrices using the Hadamard product, i.e. entrywise product. The resulting matrix is used as $\mathbf{R}_L$. Using the same parameters as in the previous example give eigenvalues and eigenvectors shown in Fig. 5a and b. The number of windows used is $I = 7$ which is a reasonable number of spectrograms to calculate. The resulting sidelobes of the TF kernel, shown in Figure 5c and d are around -35 dB both in time- and frequency domain.

4. EVALUATION

Different parameter values are chosen and the multiple windows are calculated. The number of eigenvalues needed is computed using Eq. (11). The resulting TF kernel is calculated and normalized to a highest level of 0 dB. The height of the highest sidelobe is measured outside the time-frequency area defined by $t_K = 60$ and $\omega_K = 2\pi 60/256 = 1.47$. The parameter L is varied between 5 and 40 dB. The resulting number of windows I_{case} needed as well as the highest sidelobe, given as A_{case} dB, are presented for the three different cases, case 1: only frequency sidelobe suppression, case 2: only time-domain sidelobe suppression, case 3: time- and frequency sidelobe suppression. The result is seen in Table 1. The gain of using suppression both in time and frequency is about 10 dB and the number of windows to achieve this is reasonable.

In the next evaluation the penalty function parameter L is kept constant at 20 dB and the area limited by t_K and ω_K is varied. A small area, close to the box size is difficult for the method suppressing both in time and frequency for obvious reasons, the demand on the function is impossible to fulfill. In the case of $t_K/\omega_K = 30/0.74$ the third method use just one window achieving the same suppression as where the other methods use 13 and 24 windows respectively.

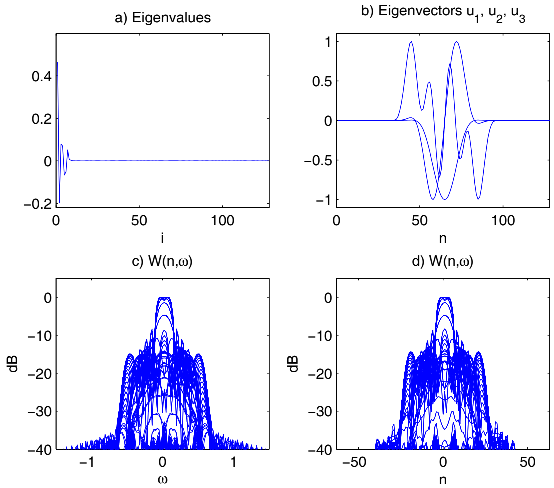

Fig. 5. a) Eigenvalues, b) The first three eigenvectors, c) The resulting TF kernel from the frequency axis, d) The resulting kernel from the time axis

$10\log_{10}(L)$	I_1/A_1	I_2/A_2	I_3/A_3
5	68/-23	74/-22	56/-21
10	44/-25	59/-21	33/-25
15	29/-24	50/-21	17/-29
20	25/-23	45/-21	10/-32
25	24/-24	44/-24	8/-30
30	23/-23	43/-24	7/-36
35	22/-23	43/-24	7/-32
40	21/-24	43/-24	7/-34

Table 1. Number of windows and sidelobe suppression for different values of L for the different cases

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5. CONCLUSIONS

A novel way of suppressing sidelobes outside a predetermined time- and frequency area of a TF kernel is presented. The method combine one matrix for frequency sidelobe supression and one matrix for time domain suppression using the Hadamard product. The advantage is that multiple windows with suppressed sidelobes can be used to calculate the TF estimate and that only a few spectrograms need to be calculated and summed.

t_K/ω_K	I_1/A_1	I_2/A_2	I_3/A_3
10/0.24	20/-5.6	22/-8.4	21/-1.7
20/0.49	9/-10	18/-11	16/-4.5
30/0.74	13/-17	24/-17	1/-16
40/0.98	17/-16	31/-17	4/-29
50/1.23	21/-17	38/-18	7/-30
60/1.47	25/-23	45/-21	10/-32
70/1.72	29/-27	49/-21	15/-31
80/1.96	31/-28	56/-22	20/-29

Table 2. Number of windows and sidelobe suppression for different values of t_K and ω_K for the different cases

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