Fractional Fourier Transforms and Wigner Distribution Functions for Stationary and Non-Stationary Random Process

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ABSTRACT

In this paper, we discuss the relations among the random process, the Wigner distribution function, the ambiguity function, and the fractional Fourier transform (FRFT). We find many interesting properties. For example, if we do the FRFT for a stationary process, although the result in no longer stationary, the amplitude of its covariance function is still independent of time. Moreover, for the FRFT of a stationary random process, the ambiguity function will be a radiant line passing through (0, 0) and the Wigner distribution function will be invariant along a certain direction. We also define the fractional stationary random process and find that a non-stationary random process can be expressed a summation of fractional stationary random processes. The proposed theorems will be useful for filter design, noise synthesis and analysis, system modeling, and communication.

1. INTRODUCTION

The Wigner distribution function (WDF) [2] is defined as:

$$W_g(t,\omega)=1/2\pi\cdot\int_{-\infty}^{\infty} g(t+\tau/2)\cdot g^*(t-\tau/2)\cdot e^{-j\omega\tau}\cdot d\tau\,. \quad (1)$$

The ambiguity function (AF) is defined as

$$A_g(\eta,\tau)=1/2\pi\cdot\int_{-\infty}^{\infty} g(t+\tau/2)\cdot g^*(t-\tau/2)\cdot e^{-jt\eta}\cdot dt\,. \quad (2)$$

The WDF and the AF are alike. They have the relation as

$$A_g(\eta,\tau)=IFT_{\omega\to\tau}\{FT_{t\to\eta}[W_g(t,\omega)]\}. \quad (3)$$

The fractional Fourier transform (FRFT) is defined as [4]:

$$G_\alpha(u)=\sqrt{\frac{1-j\cot\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{\frac{j}{2}u^2\cot\alpha-jut\csc\alpha+\frac{j}{2}t^2\cot\alpha}\,g(t)\,dt\,. \quad (4)$$

It is a generalization of the Fourier transform (FT). (When $\alpha = \pi/2$, the FRFT becomes the FT). When α=0, it becomes the identity operation. The FRFT can be applied for filter design, pattern recognition, optics analysis, communication, watermark, etc.

The relations between the FRFT and the WDF / AF have been derived in [1][3].

$$W_{G_\alpha}(u,v)=W_g(u\cos\alpha-v\sin\alpha,\ u\sin\alpha+v\cos\alpha)\,. \quad (5)$$

$$A_{G_\alpha}(\eta,\tau)=A_g(\eta\cos\alpha+\tau\sin\alpha,-\eta\sin\alpha+\tau\cos\alpha)\,. \quad (6)$$

where $W_g(u, v)$ and $W_{G\alpha}(u, v)$ are the WDFs of $g(t)$ and $G_\alpha(u)$, and $A_g(\eta, \tau)$ and $A_{G\alpha}(\eta, \tau)$ are the AFs of $g(t)$ and $G_\alpha(u)$. Thus the FRFT corresponds to the counterclockwise rotation in the WDF plane and the clockwise rotation in the AF plane.

In this paper, we discuss the relation between the FRFT and the WDF for the case where $g(t)$ is a random process. We also use the result to define the "fractional stationary process".

We often use the covariance function $R_g(u, \tau)$ and the power spectral density (PSD) $S_g(u, \omega)$ to express the statistical properties of a random process [5]:

$$R_{G_0}(u,\tau)=R_g(u,\tau)=E[g(u+\tau/2)g^*(u-\tau/2)]. \quad (7)$$

$$S_{G_0}(u,\omega)=S_g(u,\omega)=\int_{-\infty}^{\infty}R_g(u,\tau)e^{-j\omega\tau}d\tau \quad (8)$$

where E means the "expected value". We set $R_{G0}(u, \tau) = R_g(u, \tau)$ and $S_{G0}(u, \tau) = S_g(u, \tau)$ to compare the result in Sec. 2. Note that, when α = 0, the FRFT becomes the identity operation, i.e., . $g(t) = G_0(u)$.

When the random process is stationary, i.e., the statistical properties do not change with u, we can simplify (7) and (8) as:

$$R_{G_0}(\tau)=R_g(\tau)=E[g(u+\tau/2)g^*(u-\tau/2)] \text{ for any } t, \quad (9)$$

$$S_{G_0}(\omega)=S_g(\omega)=\int_{-\infty}^{\infty}R_g(\tau)e^{-j\omega\tau}d\tau\,. \quad (10)$$

2. THE FRACTIONAL FOURIER TRANSFORM FOR A STATIONARY RANDOM PROCESS

Suppose that $g(t)$ is a stationary random process and $G_\alpha(u)$ is the FRFT of $g(t)$. Then the covariance function of $G_\alpha(u)$ is:

$$R_{G_\alpha}(u,\tau)=E[G_\alpha(u+\tau/2)\,G_\alpha^*(u-\tau/2)] \quad \text{(apply (4))}$$

$$=\sqrt{\frac{1+\cot^2\alpha}{4\pi^2}}\,e^{\frac{j}{2}\cot\alpha\left[(u+\frac{\tau}{2})^2-(u-\frac{\tau}{2})^2\right]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j\left(u+\frac{\tau}{2}\right)t\csc\alpha}$$

$$\cdot e^{j\left(u-\frac{\tau}{2}\right)t_1\csc\alpha}\,e^{\frac{j}{2}\left(t^2-t_1^2\right)\cot\alpha}\,E[g(t)g^*(t_1)]\,dt\,dt_1\,. \quad (11)$$

Note that $E[g(t)g^*(t_1)] = R_g(t - t_1)$. Then we set

$$t_2 = t + t_1, \qquad t_3 = t - t_1, \quad (12)$$

and (11) can be rewritten as:

$$R_{G_\alpha}(u,\tau)=\frac{1}{2\pi|\sin\alpha|}e^{ju\tau\cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-jut_3\csc\alpha}e^{-j\frac{\tau}{2}t_2\csc\alpha}$$

$$\cdot e^{jt_2t_3\cot\alpha/2}R_g(t_3)\,dt_2\,dt_3/2\,. \quad (13)$$

From the fact that

$$\int_{-\infty}^{\infty}e^{\frac{j}{2}\cot\alpha\left(t_3-\frac{\tau}{\cos\alpha}\right)t_2}dt_2=4\pi|\tan\alpha|\,\delta\!\left(t_3-\frac{\tau}{\cos\alpha}\right), \quad (14)$$

equation (13) can be simplified as

$$R_{G_\alpha}(u,\tau)=\frac{e^{ju\tau\cot\alpha}}{|\cos\alpha|}e^{-j\frac{u\tau}{\sin\alpha\cos\alpha}}R_g\!\left(\frac{\tau}{\cos\alpha}\right). \quad (15)$$

After further simplifying (15), we obtain the following theorem:
[Theorem 1]: If $G_\alpha(u)$ is the FRFT of a stationary random process $g(t)$, then the covariance functions of $G_\alpha(u)$ and $g(t)$ have the following relation:

$$R_{G_\alpha}(u,\tau)=\frac{1}{|\cos\alpha|}e^{-ju\tau\tan\alpha}R_g\left(\frac{\tau}{\cos\alpha}\right). \tag{16}$$

Note that
(a) The amplitude of $R_{G\alpha}(u, \tau)$ is independent of u. It is only dependent on τ. Thus $G_\alpha(u)$ is nearly to be stationary.
(b) Moreover, in the case where $g(t)$ is real, since $R_g(\tau)$ is also real, we can conclude that

$$\arg\left[R_{G_\alpha}(u,\tau)\right]=-u\,\tau\tan\alpha\,. \tag{17}$$

In this case, we can use the phase of $R_{G\alpha}(u, \tau)$ to estimate the parameter α of the FRFT.
(c) When $\alpha = 0$, (16) becomes $R_g(\tau)$.
[Theorem 2] If $G_\alpha(u)$ is the FRFT of a stationary random process $g(t)$, then the power spectral densities of $G_\alpha(u)$ and $g(t)$ have the following relation:

$$S_{G_\alpha}(u,\omega)=S_g(\omega\cos\alpha+u\sin\alpha). \tag{18}$$

It can be proven from (8), (16), and the scaling and modulation properties of the FT. Thus $S_{G\alpha}(u, \omega)$ is a scaling and shifting version of $S_g(u, \omega)$ and the amount of shifting grows with u.

3. FRACTIONAL STATIONARITY

Then we define the "fractional stationary" random process. For a random process, if the statistical property of $G_\alpha(u)$ (the FRFT with order α for $g(t)$) is independent of u:

$$E\left[G_\alpha(u_1+\frac{\tau}{2})G_\alpha^*(u_1-\frac{\tau}{2})\right]=E\left[G_\alpha(u_2+\frac{\tau}{2})G_\alpha^*(u_2-\frac{\tau}{2})\right]$$
$$=R_{G_\alpha}(\tau) \qquad \text{for any } u_1 \text{ and } u_2, \tag{19}$$

we call it the **α order fractional stationary random process**. Note that the original stationary random process is the zero order fractional random process. The FRFT of order α for a stationary random process is an $-\alpha$ order fractional stationary random process. In fact,
[Theorem 3]: The FRFT with parameter β for an α order stationary random process is an $(\alpha-\beta)$ order fractional stationary random process.

We have known that the wave propagation in the optical system consisting of lenses [6] and in the graded-index fiber [7] can be expressed by the FRFT. Even if in the free space, when there is some impurities or the density of gaseous matter is not uniform, the free space will become a non-uniform medium and we may use the FRFT together with scaling and chirp multiplication to express it. A stationary random process will become a fractional stationary random process after propagating through these types of medium. Due to these reasons, in nature less of the random process is pure stationary. Instead, we may express them as the fractional stationary random process or a summation of fractional stationary random processes, see section 5.

4. WDF AND AF FOR STATIONARY AND FRACTIONAL STATIONARY RANDOM PROCESSES

Suppose that $g(t)$ is a random process. Its WDF is [9][10][11]

$$E\left[W_g(t,\omega)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}E\left[g(t+\tau/2)g^*(t-\tau/2)\right]\cdot e^{-j\omega\tau}\cdot d\tau \tag{20}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_g(t,\tau)\cdot e^{-j\omega\tau}\cdot d\tau=\frac{1}{2\pi}S_g(t,\tau).$$

If $g(t)$ is a stationary process, i.e., the random process whose statistical properties do not change with time, then (9) become

$$E\left[W_g(t,\omega)\right]=S_g(\omega)/2\pi\,. \tag{21}$$

In other words, the WDF of a stationary process is independent of t and **the slicing of the WDF along ω-axis is just the power spectral density** of the stationary process.

Moreover, the AF of $g(t)$ is

$$E\left[A_g(\eta,\tau)\right]=1/2\pi\cdot\int_{-\infty}^{\infty}E\left[g(t+\tau/2)\cdot g^*(t-\tau/2)\right]\cdot e^{-jt\eta}\cdot dt$$
$$=1/2\pi\cdot\int_{-\infty}^{\infty}R_g(t,\tau)\cdot e^{-jt\eta}\cdot dt\,. \tag{22}$$

In the case where $g(t)$ is stationary,

$$E\left[A_g(\eta,\tau)\right]=1/2\pi\cdot\int_{-\infty}^{\infty}R_g(\tau)e^{-jt\eta}\,dt=\delta(\eta)R_g(\tau)\,. \tag{23}$$

That is, $A_g(\eta, \tau) = 0$ when $\eta \neq 0$. **The AF of a stationary process is a line along η-axis**.

Moreover, from (21) and (23), we obtain that
[Theorem 4]: If $G_\alpha(u)$ The FRFT of a stationary random process $g(t)$, then its WDF and AF are

$$E\left[W_{G_\alpha}(u,v)\right]=S_g(u\sin\alpha+v\cos\alpha)/2\pi\,, \tag{24}$$
$$E\left[A_{G_\alpha}(\eta,\tau)\right]=\delta(\eta\cos\alpha+\tau\sin\alpha)R_g(-\eta\sin\alpha+\tau\cos\alpha)\,. \tag{25}$$

They can be proven from (5) and (6) or from substituting (16) and (18) into (20) and (22).

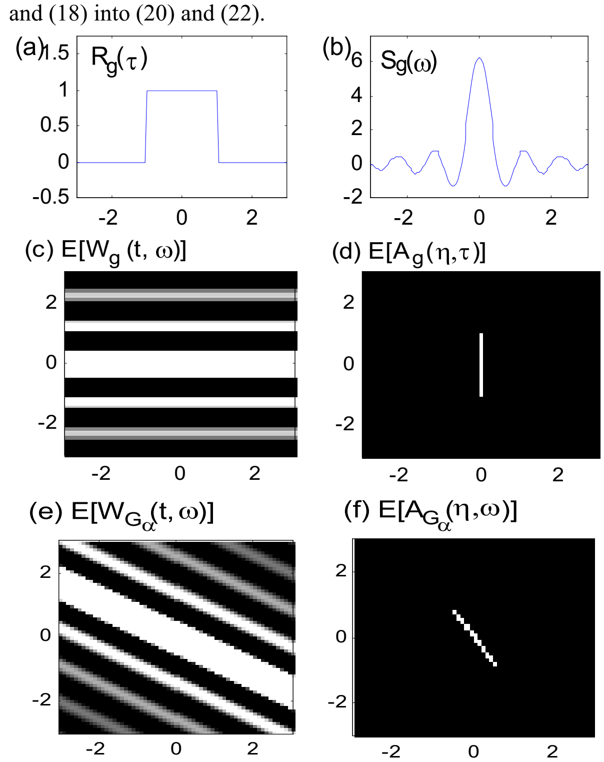

Fig. 1 The WDF and the AF for a stationary random signal whose covariance function is $R_g(\tau)$ = rect(τ/2). In (e)(f), we do the FRFT with $\alpha = \pi/6$ for the stationary random signal and find its WDF and AF.

In Fig. 1, we show an example to plot the WDF and the AF for a stationary random process whose covariance function is

$R_g(\tau) = \mathrm{rect}(\tau/2)$ (26)

where $\mathrm{rect}(t/2B) = 1$ if $t \le |B|$ and $\mathrm{rect}(t/2B) = 0$ if $|t| > B$. Its power spectral density is:

$S_g(\omega) = 2\pi\cdot\mathrm{sinc}(2\omega) = \sin(2\pi\omega)/\omega.$ (27)

Moreover, from Theorem 4 and the fact that the α order fractional stationary random process can be viewed as the FRFT with parameter $-\alpha$ for a stationary random process, we can conclude that

[Theorem 5]: If $g(t)$ is an α order fractional stationary random process, then

(a) Its WDF $E[W_g(t, \omega)]$ is invariant along the direction of

$\mathbf{d_1} = (\cos\alpha, \sin\alpha).$ (28)

That is, $W_g(t_1, \omega_1) = W_g(t, \omega)$ if $(t_1, \omega_1) = (t, \omega) + k(\cos\alpha, \sin\alpha)$.

(b) Its AF $E[A_g(t, \omega)]$ is zero except for the case where

$(t, \omega) = c\cdot\mathbf{d_2}$ where $\mathbf{d_2} = (\sin\alpha, \cos\alpha)$ (29)

Note that $\mathbf{d_1}$ rotates with α in the counterclockwise direction and $\mathbf{d_2}$ rotates with α in the clockwise direction. $\mathbf{d_1}$ and $\mathbf{d_2}$ are symmetry respect to the line of $t = \omega$.

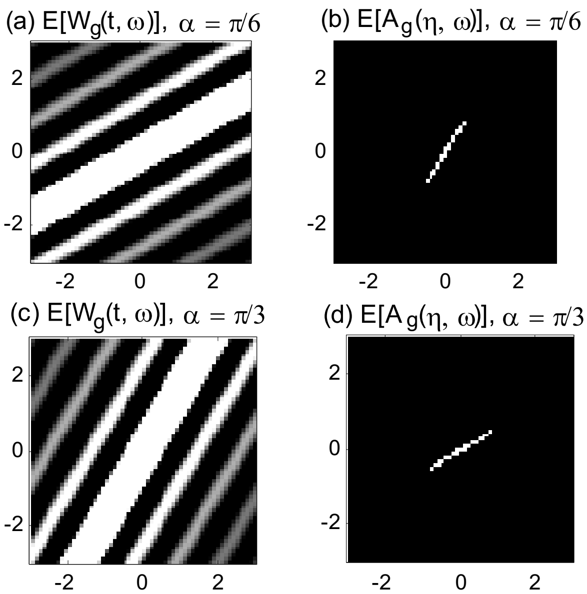

Fig. 2 The WDF and the AF for a fractional stationary random process of order α. Its fractional covariance function is $R_{G\alpha}(\tau) = \mathrm{rect}(\tau/2)$ and $\alpha = \pi/6$ in (a)(b), $\alpha = \pi/3$ in (c)(d).

In Fig. 2, we show the WDF and the AF for the fractional stationary random process of order α where

$R_{G_\alpha}(\tau) = rect(\tau/2)$. ($R_{G_\alpha}(\tau)$ is defined in (19)), (30)

Then we discuss the case where $h(t)$ is a summation of fractional stationary random processes:

$$h(t) = g_1(t) + g_2(t) + g_3(t) + \cdots\cdots + g_k(t) \quad (31)$$

where $g_n(t)$ is an $\alpha(n)$ order stationary random process. Suppose that $g_n(t)$'s are mutually independent:

$$E\left[g_m(t+\tau/2)g_n^*(t-\tau/2)\right] = 0 \quad (32)$$

for all t's and τ's if $m \neq n$,

then we can prove that

$$E\left[h(t+\tau/2)h^*(t-\tau/2)\right] = \sum_{n=1}^{k} E\left[g_n(t+\tau/2)g_n^*(t-\tau/2)\right]$$

$$= \sum_{n=1}^{k} R_{G_{n,\alpha(n)}}(t,\tau). \quad (33)$$

In other words, if we calculate its WDF and AF, we obtain

$$E\left[W_h(t,\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_h(t,\tau)\cdot e^{-j\omega\tau}\cdot d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{k} R_{G_{n,\alpha(n)}}(t,\tau)\cdot e^{-j\omega\tau}\cdot d\tau = \sum_{n=1}^{k} E\left[W_{g_n}(t,\omega)\right], \quad (34)$$

Similarly,

$$E\left[A_h(\eta,\tau)\right] = \sum_{n=1}^{k} E[A_{g_n}(\eta,\tau)]. \quad (35)$$

[Theorem 6]: If $g_n(t)$'s are mutually independent fractional stationary random processes, then the WDF and the AF of $g_1(t) + g_2(t) + g_3(t) + \ldots.. + g_k(t)$ are just the summations of the WDFs and the AFs of $g_n(t)$'s.

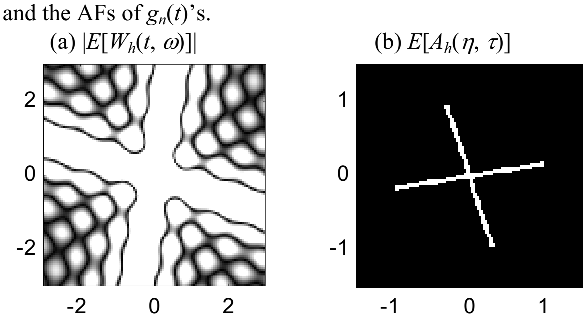

Fig. 3 The WDF and the AF of $h(t)$, where $h(t)$ is a summation of two fractional stationary random processes.

We give an example in Fig. 3, where $h(t) = g_1(t)+g_2(t)$ and $g_1(t)$ and $g_2(t)$ are -0.1π and 0.45π order fractional stationary random processes with $R_{G\alpha}(\tau) = \mathrm{rect}(\tau/2)$. Note that, $E[W_h(t, \omega)] = E[W_{g1}(t, \omega)] + E[W_{g2}(t, \omega)]$ and $E[A_h(\eta, \tau)] = E[A_{g1}(\eta, \tau)] + E[A_{g2}(\eta, \tau)]$, as the description in Theorem 6.

5. DECOMPOSITION FOR NON-STATIONARY RANDOM PROCESS

Theorems 5 and 6 lead to the following theorem:

[Theorem 7]: **Any** non-stationary random process can be expressed as a summation of mutually independent fractional stationary random processes.

(**Proof**): Suppose that $h(t)$ is a non-stationary random process. First, we calculate the expected value of the AF:

$$E\left[A_h(\eta,\tau)\right] = 1/2\pi\cdot\int_{-\infty}^{\infty} E\left[h(t+\tau/2)\cdot h^*(t-\tau/2)\right]\cdot e^{-jt\eta}dt. \quad (36)$$

Then we can decompose $E[A_g(\tau, \eta)]$ into:

$$E\left[A_h(\eta,\tau)\right] = \sum_{m=0}^{k-1} C_m(r)d_m(\theta) \quad (37)$$

where $r = (\eta^2 + \tau^2)^{1/2}$, $\theta = \arg(\eta + j\tau)$,

$C_m(r) = A_h(r\cos\alpha_m, r\sin\alpha_m)$, $\alpha_m = \pi m/k$,

$d_m(\theta) = 1$ when $\pi(m-1/2)/k < \theta \le \pi(m+1/2)/k$,

$d_m(\theta) = 0$ otherwise, $k \to \infty$. (38)

If k is very large, $d_m(\theta)$ will converge to

$$d_m(\theta) \to \frac{2\pi r}{k}\,\delta(\eta \sin\phi_m - \tau\cos\phi_m) \quad \text{when } k \to \infty. \tag{39}$$

Then (37) can be rewritten as

$$E[A_h(\eta,\tau)] \approx \sum_{m=0}^{k-1} R_m(r)\,\delta(\eta\sin\phi_m - \tau\cos\phi_m) \tag{40}$$

where $R_m(r) = 2\pi\, r A_h(r\cos\phi_m, r\sin\phi_m)/k$. (41)

Note that, from Theorem 5, $R_m(r)\delta(\eta\cdot\sin\phi_m - \tau\cdot\cos\phi_m)$ can be viewed as the AF of an $\pi/2 - \phi_m$ order fractional stationary random process. Then, together with Theorem 6, we can conclude that $h(t)$ is a summation of k mutually independent fractional stationary random processes $g_1(t)$, $g_2(t)$, …., $g_k(t)$:

$$h(t) = g_1(t) + g_2(t) + g_3(t) + \cdots\cdots + g_k(t), \tag{42}$$

$$E\left[G_{m,\phi_m}(u+\tau/2)G^*_{m,\phi_m}(u-\tau/2)\right] = 2\pi\tau A_h(\tau\cos\phi_m, \tau\sin\phi_m)/k \quad \text{(independent of } u\text{)},$$

$\varphi_m = \pi/2 - \pi m/k$. #

Although the stationary random process is popular in theory, in nature, most of the noises are non-stationary ones. Now, from Theorem 7, with the aids of the FRFT and the AF, we can decompose any non-stationary random process into a summation of fractional stationary random processes. It will be of a great help for signal processing, such as noise synthesis, system modeling, filter design, and improving the quality of signal transmission.

For example, suppose that after several times of experiments, we find that the transmitted signal $x(t)$ is interfered by a non-stationary random process $h(t)$. We also suppose that all the noise sources are stationary and $h(t)$ is the mixture of them.

First, we find the mean of the ambiguity function of $h(t)$. Then, we can use the procedure in (36)~(42) to decompose $h(t)$ into a summation of fractional stationary random processes:

$$h(t) = g_{\alpha_1}(t) + g_{\alpha_2}(t) + \cdots\cdots + g_{\alpha_k}(t) \tag{43}$$

where $g_{\alpha k}(t)$ is an α_k order fractional stationary random process, $\alpha_m = \pi m/k$, as in (38). Then $h(t)$ can be synthesized as Fig. 4,

$$\text{where } n_{\alpha_m}(t) = O_F^{\alpha_m}\left[g_{\alpha_m}(t)\right]. \tag{44}$$

Note that $n_m(t)$'s are stationary random processes. We use Fig. (4) to model $h(t)$ since

$$\begin{aligned}
& O_F^{-\alpha_1}\Big(O_F^{\alpha_1-\alpha_2}\Big\{\cdots\Big[O_F^{\alpha_{k-2}-\alpha_{k-3}}\Big(O_F^{\alpha_{k-1}-\alpha_{k-2}}\Big\{O_F^{\alpha_{k-1}-\alpha_k}[n_k(t)] \\
& \quad + n_{k-1}(t)\Big\} + n_{k-2}(t)\Big]\cdots + n_2(t)\Big\} + n_1(t)\Big) \\
& = O_F^{-\alpha_1+\alpha_1-\alpha_2+\alpha_2-\alpha_3+\cdots\cdots+\alpha_{k-1}-\alpha_k}[n_k(t)] + \\
& \quad O_F^{-\alpha_1+\alpha_1-\alpha_2+\alpha_2-\alpha_3+\cdots\cdots+\alpha_{k-2}-\alpha_{k-1}}[n_{k-1}(t)] + \cdots\cdots\cdots + \\
& \quad O_F^{-\alpha_1+\alpha_1-\alpha_2}[n_2(t)] + O_F^{-\alpha_1}[n1(t)] \\
& = O_F^{-\alpha_k}[n_k(t)] + O_F^{-\alpha_{k-1}}[n_{k-1}(t)] + \cdots\cdots\cdots + O_F^{-\alpha_2}[n_2(t)] + O_F^{-\alpha_1}[n_1(t)] \\
& = g_{\alpha_1}(t) + g_{\alpha_2}(t) + \cdots\cdots + g_{\alpha_k}(t) = h(t)
\end{aligned} \tag{45}$$

From Fig. 4, we can estimate the locations of noise sources or the distribution of the medium in the path of signal transmission. It is also helpful for us to design a series of FRFT filters [8] to filter output the noise.

6. CONCLUSIONS

We introduced the relations among the random process, the ambiguity function, and the fractional Fourier transform. We find several beautiful properties. We also defined the fractional random process. We found that any non-stationary random process can be decomposed into a summation of fractional stationary random processes. The proposed idea will be useful for communication, filter design, and system modeling.

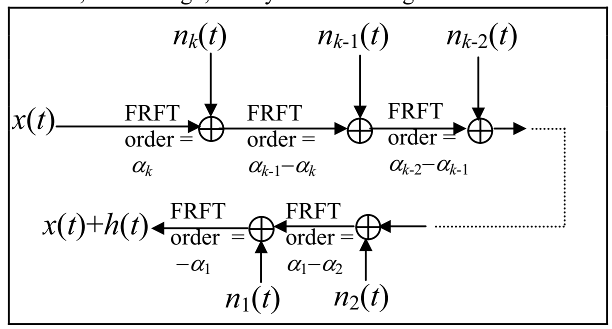

Fig. 4 Using the FRFT to model the non-stationary random process $h(t)$, where $x(t)$ is the input signal, $n_1(t)$, $n_2(t)$, …., $n_k(t)$ are stationary random processes.

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