AN EXTENSION OF THE CLASS OF UNITARY TIME–WARPING PROJECTORS TO DISCRETE–TIME SEQUENCES

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ABSTRACT

This paper establishes a new coherent framework to extend the class of unitary warping operators [1] to the case of discrete–time sequences. Providing some *a priori* considerations on signals, we show that the class of discrete–time warping operators finds a natural description in linear shift–invariant spaces. On such spaces, any discrete–time warping operator can be seen as a non–uniform weighted resampling of the original signal. Then, gathering different results from the non–uniform sampling theory, we propose an efficient iterative algorithm to compute the inverse discrete–time warping operator and we give the conditions under which the warped sequence can be inverted. Numerical examples show that the inversion error is of the order of the numerical round–off limitations after few iterations.

1. INTRODUCTION

Signal processing methods are often based on a change of the representation space. This change is generally performed by projecting the original space into another one, adapted to a particular class of signals. The underlying idea is that some spaces are better suited than others to highlight specific properties of signals. As a consequence, it is a natural feature to perform processing tasks on the projected space since the useful information is easily reachable. As a final step, a well–defined inverse projection allows to return back to the original domain. This projection–processing–inversion framework has been successfully used in various signal processing domains [2].

An interesting class of unitary projections is the class of time warping operators [1]. This class has been used in image processing [3] for non–linear coordinate transformations and morphing purposes. In signal processing, warping operators have been used to build time–frequency representations with reduced interference terms, the so–called VU–Cohen's class [1]. Despite some other applications, the reversibility property of the time warping operators has surprisingly not found signal–processing applications as is the case for other unitary transforms. Recently, we have shown that a projection–processing–inversion framework, in time–warped spaces, can be used for efficient non–stationary denoising purpose [4]. Still, because of our non–exact approach, cumulative errors led to inaccurate results in multi–stages processing.

As far as our knowledge, an extension of the class of time warping operators, while keeping in mind invertibility, has not been derived yet in the case of discrete–time signals. We believe that this lack may explain the small number of signal–processing methods based on this class of operators. As an attempt to fill this lack, this paper establishes a new coherent framework to extend the class of warping operators in the case of discrete–time sequences and derive an efficient implementation of the inverse operator as well as its invertibility conditions.

The organization of this paper is as follows. Section 2 starts with the classical definition of the class of continuous time–warping operators and describes its mathematical properties. Then a discrete–time formulation is proposed in shift–invariant spaces. Section 3 states the equivalence between inversion of the discrete–time warping operator and the inversion of a resampling operator. Gathering different results from the non–uniform sampling theory, an efficient iterative implementation is proposed, and invertibility conditions on the resampling set are derived. Numerical and convergence results are given section 4, and concluding remarks are given section 5.

2. CLASS OF UNITARY TIME–WARPING OPERATORS

2.1. Continuous formulation

Given $x(t) \in L^2(\mathbb{R})$, the set of unitary time–warping operators $\{\mathcal{W},\ w(t) \in \mathcal{C}^1,\ \dot{w}(t) \geq 0 : x(t) \to (\mathcal{W}x)(t)\}$, is defined in [1] by

$$(\mathcal{W}x)(t) = |\dot{w}(t)|^{1/2} x\left(w(t)\right), \tag{1}$$

where $\dot{w}(t)$ stands for the derivative of the warping function $w(t)$ with respect to t. Properties of this transformation include linearity and unitary equivalence since the envelope $|\dot{w}|^{1/2}$ preserves the energy in the signal at the output of $\mathcal{W}$. Because of the latter property, it is straightforward to state the existence of the inverse warping–operator

$$(\mathcal{W}^{-1}x)(t) = |\dot{w}(w^{-1}(t))|^{-1/2} x\left(w^{-1}(t)\right). \tag{2}$$

2.2. Discrete formulation

For real–life applications, the continuous formulation of the class of warping operators defined in Sec. 2.1 has to be turned into a discrete formulation. Let $x[n] \in \mathbb{R}^N$, $n = 0, \ldots, N-1$ be the sequence obtained by uniform sampling of the continuous signal, $x[n] = \int_t x(t)\delta(t - nT)dt$ with T the sampling rate, and $x_{\mathcal{W}}[m] \in \mathbb{R}^M$, $m = 0, \ldots, M-1$ be the warped discrete sequence. Since we are now dealing with finite–length sequences, we shall restrict ourself to the class of warping functions defined in the interval $[0, (N-1)T]$, for which $w(0) = 0$, and $w(nT) = nT$.

This work was supported by the French Military Center of Oceanography under the reaserch contract CA/2003/06/CMO.

For the sake of notation simplicity, we denote by $\overline{m}$ the normalized sequence $m/(M-1),\ m=1\ldots M-1$. Then a straightforward definition for the sampled discrete–time warping operators is

$$(\mathcal{W}x)[m] = |\dot{w}_d(\overline{m})|^{1/2}\, x\left(w_d(\overline{m})(N-1)T\right), \tag{3}$$

where the warping function $w_d(t)$ is defined by $\{w_d : [0,1] \rightarrow [0,1] \in \mathcal{C}^1 |\ w_d(0)=0,\ w_d(1)=1,\ \dot{w}_d(t)\geq 0\}$. From this definition, the computation of the discrete–time warping operator requires samples $x\left(w_d(\overline{m})(N-1)T\right)$. However, from the sequence $x[n]$, only samples $x(nT)$ are known and the recovery of the missing samples has to deal with this partial knowledge.

From the Shannon's theory [5], it is well known that any bandlimited signal can be exactly recovered from its uniform samples with the so–called *sinc interpolator* by

$$x(t) = \sum_n x[n]\mathrm{sinc}(t-nT), \tag{4}$$

where $\mathrm{sinc}(t) = \sin(\pi t)/\pi t$. However this method is generally not used because of the slow decay of the sinc function with order $\mathcal{O}(1/x)$ which is not well–suited for practical applications.

More powerful methods can be found in an interpolation perspective [6]. One of those is the general class of interpolators in *linear shift–invariant* spaces.

A linear shift–invariant space V_ϕ is uniquely determined by the *kernel* ϕ with $V_\phi = \mathrm{Span}(\{\phi(.-k),\ k\in\mathbb{Z}\})$. The general interpolation formula on V_ϕ is given by [7]

$$x = \sum_{k\in\mathbb{Z}} \underbrace{\Big[\sum_n x[n]\psi(k-n)\Big]}_{a_k}\phi(.-kT), \tag{5}$$

where ψ is the impulse response of some projection filters, and the coefficients (a_k) are the result of the filtering. In the scope of this paper we shall restrict ourself to the case of *exact interpolation*, which is equivalent to $x[n] = x(t)|_{t=nT},\ n=0..N-1$. The latter condition is met, in the shift–invariant space V_φ generated by the kernel $\varphi(.) = \sum_{k\in\mathbb{Z}}\psi(k-n)\phi(.-kT),\ k\in\mathbb{Z},\ t\in\mathbb{R}$, if and only if φ verifies the *exact interpolation condition*

$$\varphi(nT) = \sum_{k\in\mathbb{Z}}\psi(k-n)\phi(nT-kT) = \delta_{n,0},\ \forall n\mathbb{Z}, \tag{6}$$

where $\delta_{n,m}$ denotes the Kronecker delta function. This interpolation method allows a degree of freedom on the choice of the interpolation kernel $\varphi(t)$. In truth, this choice is a matter of *a priori* considerations on the signal $x(t)$. If one deals with bandlimited signals then $\varphi(t) = \mathrm{sinc}(t)$ has to be chosen to recover Equ. 4. On the other hand, if the signal $x(t)$ can be modelled by a spline, then the cardinal B–spline [7] is the optimal choice.

Let $\mathcal{S} : \{x[n]\} \rightarrow x_{\mathcal{S}}[m],\ n=0..N-1,\ m=0..M-1$ be the resampling operator on the shift–invariant space V_φ defined by

$$x_{\mathcal{S}}[m] = (\mathcal{S}x)[m] = \sum_n x[n]\varphi(f(m)-n), \tag{7}$$

for some resampling mapping f. Defining $n_m = f(m) = (N-1)\ w(\overline{m})$, the set $\mathcal{X} = \{n_m\},\ m=0..M-1$ is a non–uniform sampling set for the V_φ space. This gives the final expression for the class of discrete–time warping operators

$$x_{\mathcal{W}}[m] = (\mathcal{W}x)[m] = |\dot{w}_d(\overline{m})|^{1/2}\ x_{\mathcal{S}}[m], \tag{8}$$

which can be seen as a weighted resampling in V_φ of the sequence $x[n]$.

3. DISCRETE-TIME INVERSE WARPING OPERATOR

3.1. Problem statement

Our starting point is the definition of the discrete-time inverse warping operator $\mathcal{W}^{-1}$

$$(\mathcal{W}^{-1}(\mathcal{W}x))[n] \triangleq x[n]. \tag{9}$$

Then, defining $\mathcal{S}^{-1}$ the inverse sampling operator, and using Equ. 8 leads to

$$(\mathcal{W}^{-1}x_{\mathcal{W}})[n] = (\mathcal{S}^{-1}\,|\dot{w}_d(\overline{m})|^{-1/2}\,x_{\mathcal{W}}[m])[n]. \tag{10}$$

Inversion of the discrete-time warping operator resumes to the inversion of the sampling operator which is a difficult task in shift–invariant spaces for any kernel function.

3.2. Equivalence in non–uniform sampling theory

The problem of recovering a signal $x \in V$ from a non–uniformly distributed set of samples is generally referred as a non–uniform sampling problem [8].

It can be shown that if the maximal gap between the samples $n_m,\ n_{m+1}$, is small enough, then any $x \in V_\varphi$ can be recovered from the set $\{x_{\mathcal{S}}[m]\}$, and one says that the sampling set $\mathcal{X}$ is stable in V_φ. Conditions on $\mathcal{X}$ to be stable in V_φ are discussed in Sec. 3.3. Then, from [8] and [9] we derive the following iterative algorithm of the inverse sampling operator.

Alg. 1 (Inverse sampling operator)**.** *Let $\varphi(.)$ be a kernel for the shift–invariant space V_φ. For all $\varphi(t)$ verifying*

$$\sum_n \sup_{t\in[0,1]} |\varphi(t-n)| < \infty,\ \forall\, n\in\mathbb{Z},\ t\in\mathbb{R}, \tag{11}$$

$$\varphi(t)|_{t=nT} = \delta_{n,0},\ \forall n\in\mathbb{Z},\ t\in\mathbb{R}, \tag{12}$$

and providing $\mathcal{X} = \{n_m\},\ m=0..M-1$ a stable sampling set in V_φ, the uniform samples $x[n],\ n=0..N-1$ for all $x\in V_\varphi$ can be recovered by the following iterative algorithm.

- Initialization

  $$x^{(0)}[n] = x_{\mathcal{S}}[k],\ k = \underset{m}{\mathrm{argmin}}\,\{|n-n_m|\}$$

  $$x_{\mathcal{S}}^{(0)}[m] = \sum_{n=0}^{N-1} x^{(0)}[n]\varphi(n_m-n)$$

- Until $\|x^{(p)} - x^{(p-1)}\|_2 < \varepsilon$ do

  $$\Delta x^{(p)}[m] = x_{\mathcal{S}}[k] - x_{\mathcal{S}}^{(p-1)}[k],\ k = \underset{m}{\mathrm{argmin}}\,\{|n-n_m|\}$$

  $$x^{(p)}[n] = x^{(p-1)}[n] + \Delta x^{(p)}[n]$$

  $$x_{\mathcal{S}}^{(p)}[m] = \sum_{n=0}^{N-1} x^{(p)}[n]\varphi(n_m-n)$$

- End

and $\lim_{p\to\infty}\left\|x[n]-x^{(p)}[n]\right\|_2 = 0$ *with a geometric convergence.*

3.3. Maximal gap between samples

It is obvious that a signal $x \in V_\varphi$ is not always uniquely determined for all sampling set $\mathcal{X} = \{n_m\}$, $m = 0..M-1$, especially if $\mathcal{X}$ contains large gaps. In the case of bandlimited function the Beurling–Landau's theorem [10] provides a condition on $\mathcal{X}$ to be stable. However, in the case of shift–invariant spaces, this result does not hold anymore and the exact conditions on $\mathcal{X}$ to be stable in V_φ are unknown so far. Recently, under–optimal stability conditions have been determined for shift–invariant spaces in [9].

Let B_m be the δ–*ball* defined by

$$B_m = \{x : |n_m - x| \leq \delta\},\ x \in [0, N-1]. \tag{13}$$

We define the *maximal gap* the smallest δ such that

$$\bigcup_m B_m = [0, N-1]. \tag{14}$$

Then it can be shown that the upper bound

$$\delta < \left\| \frac{\pi\ G_\varphi(\omega)}{T\ G_{\dot\varphi}(\omega)} \right\|_0, \tag{15}$$

guarantees the sampling set $\mathcal{X}$ to be stable in V_φ. The functions $G_\varphi(\omega)$ and $G_{\dot\varphi}(\omega)$ are both related to the Fourier transform $\hat\varphi(\omega)$ of the kernel function $\varphi(t)$ by

$$G_\varphi(\omega) = \left(\sum_k |\hat\varphi(\omega + 2k\pi)|^2 \right)^{1/2}, \tag{16}$$

$$G_{\dot\varphi}(\omega) = \left(\sum_k |j\omega\ \hat\varphi(\omega + 2k\pi)|^2 \right)^{1/2}. \tag{17}$$

Because $G_\varphi(\omega)$ and $G_{\dot\varphi}(\omega)$ are both 2π–periodic, the norm $\|.\|_0$ is given by $\|G_.(\omega)\|_0 = \inf_{\omega \in [0,2\pi]} G_.(\omega)$.

Since the maximal gap is equal to $\sup_m |n_{m+1} - n_m|/2$, it is easy to show that

$$\sup_m |n_{m+1} - n_m| \leq \sup_{t\in[0,1]}(\dot w_d(t))\ \frac{2\,(N-1)}{M-1} \leq \delta, \tag{18}$$

and to establish the under–optimal stability condition on M

$$M > 2\,(N-1) \sup_{t\in[0,1]}(\dot w_d(t)) \left\| \frac{\pi\ G_\varphi(\omega)}{T\ G_{\dot\varphi}(\omega)} \right\|_0^{-1} + 1 > N. \tag{19}$$

Then for any sequence $\mathcal{W}x[m]$, conditions under which the discrete–time warping operator can be inverted only depend on the kernel function and the maximum of the derivative of the warping function.

4. EXPERIMENTAL RESULTS

We illustrate, in this section, our method on a numerical example. We consider here the shift–invariant space V_{φ^a} generated by

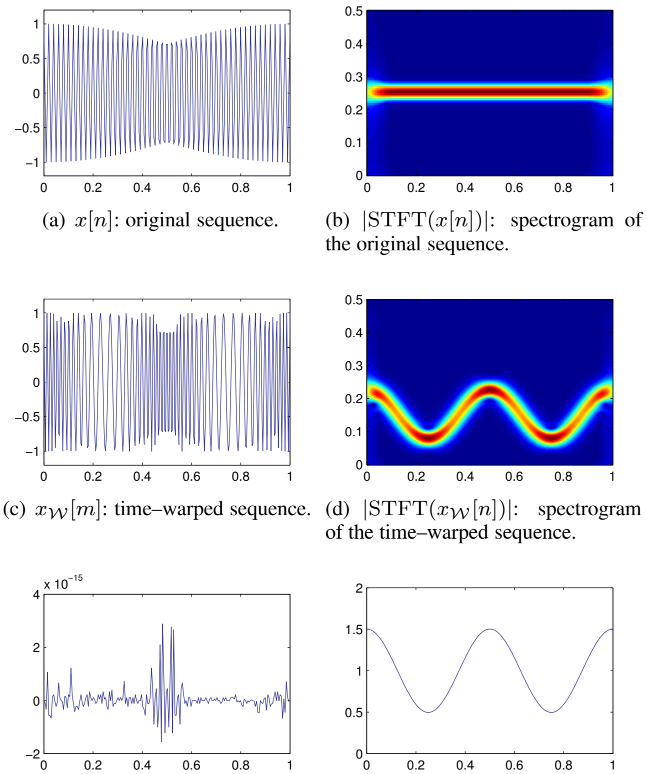
$$\varphi^a(t) = \mathrm{sinc}(t)\cos\left(\frac{\pi t}{2a}\right)^2 \Pi_{[-a,a]}(t), \tag{20}$$

where the function $\Pi_{[-a,a]}(t) = 1,\ t < |a|,\ 0$ otherwise. $\varphi^a(t)$ has to be seen as an approximation of the sinc function in the sense that $\lim_{a\to\infty} \varphi^a(t) = \mathrm{sinc}(t)$. This kernel belongs to the class of windowed–sinc interpolators [3] and is generally preferable to the sinc function since it has a compact support and leads to a reduction of the ringing artifacts.

For any $a < \infty$ it is obvious that $\varphi^a(t)$ verify Equ. 11 and Equ. 12, and the iterative algorithm always converges for a stable set $\mathcal{X}$.

The sequence $x[n] = \cos(2\pi 50 \overline{n}),\ n = 0, \ldots, 199$ is first generated. The discrete–time warping operator we use is defined by the warping function $w_d(t) = t + 0.04\ \sin(4\pi t)$. The warped sequence $x_\mathcal{W}[m],\ m = 0, \ldots, 319$ is generated with $\varphi^5(t)$ by means of Equ. 8. Then we use Equ. 10 and Alg. 1 to recover the original sequence $x[n]$. Results of the numerical simulation are depicted in Fig. 1.

Fig. 1(a) shows the original discrete–time cosine sequence and Fig. 1(b) its time–frequency representation. Fig. 1(c) shows the warped sequence and Fig. 1(d) its time–frequency representation. The instantaneous frequency of the warped sequence is cosine modulated. This non–linear modulation effect comes from the derivative of the warping function represented Fig 1(f). Fig. 1(e) shows the difference between the original sequence and the sequence recovered with the inverse–warping operator after 45 iterations. As seen, the maximal reconstruction error is of the order of the round–off precision we used ($\varepsilon \approx 2.22\ 10^{-16}$ in our example).



(a) $x[n]$: original sequence.
(b) $|\mathrm{STFT}(x[n])|$: spectrogram of the original sequence.
(c) $x_\mathcal{W}[m]$: time–warped sequence.
(d) $|\mathrm{STFT}(x_\mathcal{W}[n])|$: spectrogram of the time–warped sequence.
(e) $x[n] - \left(\mathcal{W}^{-1}\left(\mathcal{W}x\right)\right)[n]$: reconstruction error.
(f) $\dot w_d(t)$: Derivative of the warping function.

Fig. 1. Numerical example of discrete–time warping operators. In this example $N = 200$, $M = 320$ and $a = 5$. (a)(b) original cosine sequence. (c)(d) time–warped sequence. (e) reconstruction error between the original sequence and the sequence recovered from the warped sequence after 45 iterations. (f) derivative of the warping function.

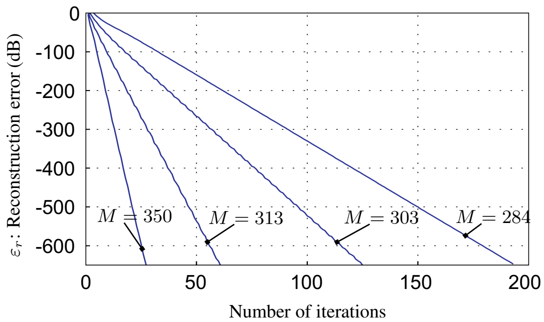


Fig. 2. Reconstruction error versus number of iterations for different sizes of resampling sets ($M = 284, 303, 313, 350$).

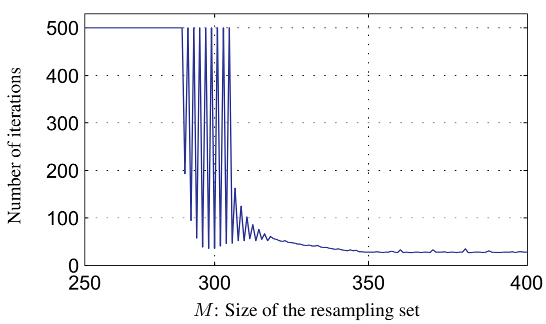


Fig. 3. Number of iterations necessary to reach $\varepsilon_r < -320$ dB, versus the size of the resampling set. A number of iterations equal to 500 signifies that the iterative algorithm does not converge for the current resampling set

Fig. 2 and Fig. 3 show results of convergence. Fig 2 shows the reconstruction error $\varepsilon_r = 20\ \log \|x[n] - \mathcal{W}^{-1}(\mathcal{W}x)[n]\|_2$ as a function of the number of iterations, for different sizes of resampling sets. Clearly, the reconstruction error is linearly decreasing on a dB scale as the iterations increase. This confirms the geometric convergence of the inverse sampling algorithm stated in Alg. 1.

As can be seen in Fig. 2, the size of the resampling set is critical as regards of the number of iterations needed to reach a fixed reconstruction error bound. As an example, one needs 10 times more iterations for a resampling set with size $M = 284$ than for a set with a size $M = 350$. This is an expected result since it is well-known that the repartition of the sampling set is related to the conditioning of the non–uniform sampling problem, and so to the convergence rate of the iterative reconstruction algorithm.

Fig 3 shows the number of iterations needed to reach the error bound $\varepsilon_r < -640$ dB, as a function of the size of the resampling set. In this example, a number of iterations equal 500 iterations signifies that the iterative algorithm does not converge for the current resampling set. Below $M = 280$ the resampling set is not stable and the iterative algorithm does not converge. Between $M = 280$ and $M = 318$ the resampling set is *critically stable* and a small perturbation of a stable set may give an unstable set. After $M = 318$, the resampling set is stable and the number of iterations needed to reach the fixed error bound is globally decreasing. This result speaks in favour of large values for M for practical applications. However, the size of the resampling set cannot be set as large as wanted for computation burden reasons and a trade–off has to be found between converging rate and computation cost.

5. CONCLUSION

We have established a new coherent framework to extend the class of warping operators to the case of discrete–time sequences and defined conditions under which such operators are invertible.

We have first considered the original discrete signal as a sampling procedure in a shift–invariant space and shown that any discrete–time warping operator can be written as a weighted resampling of the original signal.

Before giving stability conditions on the resampling set, we have shown that any discrete–time warping operator can be inverted by an efficient iterative algorithm with geometric convergence.

Finally, we have illustrated performances of the method on numerical examples and showed that the error of reconstruction of the inverse discrete–time warping operator is of the order of the round–off precision after few iterations.

We have already shown that time–warped spaces, can be used for efficient non–stationary denoising purpose [4]. We think that this new definition of the class of discrete–time warping operators can be useful for multi–stages signal denoising algorithms and separation of components with non–linear instantaneous frequency laws. These are some of the issues we intend to consider in details in future works.

6. REFERENCES

[1] R. Baraniuk, “Unitary equivalence: A new twist on signal processing,” *IEEE Trans. on Signal Processing*, vol. 43, no. 10, pp. 2269–2282, Oct. 1995.

[2] R. Allen and D. Mills, *Signal Analysis: time, frequency, scale, and structure.*, Wiley–IEEE Press, 2003.

[3] G. Wolberg, *Digital image warping*, IEEE computer society press, 1990.

[4] A. Jarrot, C. Ioana, and A. Quinquis, “Denoising underwater signals propagating through multi–path channels,” in *Oceans’05 Europe, Brest, France*, June 2005.

[5] C. A. Shannon, “A mathematical theory of communication,” *the bell system technical journal*, vol. 27, pp. 379–423 and 623–656, 1948.

[6] E. Meijering, “A chronology of interpolation: from ancient astronomy to modern signal and image processing,” *Proceedings of the IEEE*, vol. 90, pp. 319–342, Mar. 2002.

[7] P. Thévenaz, T. Blu, and M. Unser, “Interpolation revisited,” *IEEE Trans. on medical imaging*, vol. 19, no. 7, pp. 739–758, July 2000.

[8] A. Aldroubi and K. Gröchenig, “Non–uniform sampling and reconstruction in shift–invariant spaces,” *SIAM Review*, vol. 43, no. 4, pp. 585–620, 2001.

[9] W. Chen and J. Rong-Quing, “Maximal gap of a sampling set for the exact iterative reconstruction algorithm in shift invariant spaces,” *IEEE Signal Processing Letters*, vol. 1, no. 11, Nov. 2004.

[10] A. Beurling, “The collected work of a. beurling,” in *Carelson et al Eds.*, vol. 2, pp. 341–365. Birkauser, Boston, 1989.