# DOUBLE PRECONDITIONING FOR THE GABOR FRAME OPERATOR


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## ABSTRACT

We present an application of the idea of preconditioning in the context of Gabor frames. We propose a method to find an approximation for the inversion of the Gabor frame operator, based on (double) preconditioning. We thereby obtain very good approximations of the true dual Gabor atom at very low computational costs. Part of the efficiency of the proposed scheme results from the fact that all the matrices involved share a well-known block matrix structure. For Gabor atoms typically used in applications, the combination of these two preconditioners leads to very good results.


## 1. INTRODUCTION:

The Short-time Fourier transform (STFT), also called Gabor-Transform in its sampled variant, is a well known, valuable tool for displaying the energy distribution of a signal $f$ over the time-frequency plane. An important question is, how to find a Gabor analysis-synthesis system with perfect (or satisfactorily accurate) reconstruction in a numerical efficient way. Basic Gabor theory [1] states that for Gabor frames, when using the so called canonical *dual* Gabor atom $\tilde{g} = S^{-1}g$, perfect reconstruction is always achieved. For calculation the Neumann algorithm with relaxation parameter $\lambda$ can be applied. If the inequality $\|Id - \lambda S\|_{Op} < 1$ holds, this algorithm converges and the algorithm approximates the dual Gabor atom $\tilde{g}$.

For application a numerical efficient way to find this inverse is important. There are numerous iterative algorithms to invert matrices. In this work these are combined with another well known tool to speed up the convergence rate, namely, *preconditioning*. This is used to further improve the numerical efficiency of the calculation of the inverse Gabor frame matrix. The aim of this article is to investigate the idea of *double preconditioning* of the frame operator $S$. This scheme relies on the very special structure of the Gabor frame operator $S$. For $S$ let $D = (d_{i,j})$, with $d_{i,j} = \delta_{i,j} s_{i,j}$, where $\delta_{i,j}$ denotes the *Kronecker symbol*. If $S = (s_{i,j})$ is strictly diagonal dominated, it is well known that $S^{-1}$ can be approximated well using the preconditioning matrix $P = D^{-1}$. An analogue property holds if $\hat{S}$, the Fourier transformation of $S$ [2], is strictly diagonal dominated, obtaining a circulant matrix as preconditioning matrix. If using these two preconditioning matrices at the same time, hence the name *double preconditioning*, we will get a new method. For the calculation of the preconditioning matrix we use the block structure of the Gabor frame matrix, as investigated in [3], which leads to a very efficient algorithm.

This article is a short summary of results for this algorithm. For more details refer to [4].


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## 2. PRELIMINARIES AND NOTATIONS

### 2.1. Matrices

In this paper we work with complex vectors of length $n$ $x = (x_0, x_1, \ldots, x_{n-1})$, as well as with $n \times n$ matrices $A = (a_{k,l})$, symbolically denoted by $A \in M_{n,n}$. This set is equipped with the usual operator norm $\|A\|_{Op}$. By $A^*$ we denote the *adjoint* of the matrix $A$. The notion of Fourier transformation can be easily extended to matrices [5] [2] by setting $\hat{A} = F_n \circ A \circ F_n^*$, where $F_n$ is the *FFT-matrix*, $(F_n)_{k,l} = \frac{1}{\sqrt{n}} \cdot e^{-\frac{2\pi i k l}{n}}$.

Instead of solving a linear system of equations $Ax = b$ another one, $PAx = Pb$, is solved. If the matrix $P$ is chosen properly, this results in a low number of operations and small memory requirements. It can also improve the numeric stability of the system. This is called *preconditioning*[6].

### 2.2. Frames

A sequence of vectors $(g_k | k \in K) \subseteq \mathcal{H}$ is called a *frame* for the Hilbert space $\mathcal{H}$, with inner product $\langle .,. \rangle$, if constants $A, B > 0$ exist, such that

$$A \cdot \|f\|^2 \leq \sum_{k \in K} |\langle f, g_k \rangle|^2 \leq B \cdot \|f\|^2 \ \forall f \in \mathcal{H} \quad (1)$$

The constants $A$ and $B$ are called *lower frame bound* and *upper frame bound*, respectively.

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For such a sequence the operator $S : \mathcal{H} \to \mathcal{H}$ defined by $S(f) = \sum_k \langle f, g_k \rangle \cdot g_k$ is called the *frame operator*. It is self-adjoint, positive and invertible [7].

Let $(\tilde{g}_k) := \left(S^{-1} g_k\right)$. Then this is also a frame with frame bounds $B^{-1}$, $A^{-1} > 0$, the so-called *canonical dual frame*. Moreover, every $f \in \mathcal{H}$ has expansions

$$f = \sum_{k \in K} \langle f, \tilde{g}_k \rangle \, g_k \text{ and } f = \sum_{k \in K} \langle f, g_k \rangle \, \tilde{g}_k.$$

In the discrete, finite-dimensional case, $\mathcal{H} = \mathbb{C}^n$, a sequence is a frame if and only if it spans $\mathcal{H}$.

There is a number of algorithms for inverting the frame operator. A well known algorithm is the Neumann algorithm. With a special relaxation parameter $\lambda = \frac{2}{A+B}$, it becomes the so-called *frame algorithm* [7]. Its calculation requires the computation of the frame bounds, which are numerically costly to compute. In order to deal with this drawback, for example the conjugate gradient algorithm was proposed [7]. The algorithm proposed in this article avoids this drawback, as well.

### 2.3. Gabor Analysis

Recall [8] that for any non-zero window function $g$ and a signal $f$ the STFT can be defined as $\mathcal{V}_g(f)(t,\omega) = \langle f, M_\omega T_t g \rangle$ using the translation operator $T_\tau f(z) = f(z - \tau)$ and the modulation operator $M_\omega f(t) = f(t) \, e^{2\pi i \omega t}$. In $L^2(\mathbb{R}^d)$, the space of square-integrable functions from $\mathbb{R}^d$ to $\mathbb{C}$, we have

$$\mathcal{V}_g(f)(t,\omega) = \int_{\mathbb{R}^d} f(x) \overline{g(x-t)} e^{-2\pi i \omega x} dx$$

**Definition 1** *For a non-zero function g (the* window*) and parameters $\alpha, \beta > 0$, the set of time-frequency shifts of g*

$$\mathcal{G}(g, \alpha, \beta) = \{ M_{\beta n} T_{\alpha k} g : k, n \in \mathbb{Z}^d \}$$

*is called a* Gabor system. *If it is a frame, it is called a* Gabor frame.

The dual frame of a Gabor frame is a Gabor system again, which is generated by the *dual window* $\tilde{g} = S^{-1} g$ and the same parameters $\alpha$ and $\beta$.

#### 2.3.1. Discrete Gabor Analysis

From now on all vectors in the Hilbert space $\mathbb{C}^n$ are considered to be periodic. The modulation and time shift operators are discretized and periodized, i.e.,

$$T_l x = (x_{n-l}, x_{n-l+1}, \ldots, x_0, x_1, \ldots, x_{n-l-1})$$

and $M_k x = \left( x_0 \cdot W_n^0, x_1 \cdot W_n^{1 \cdot k}, \ldots, x_{n-1} \cdot W_n^{(n-1)k} \right)$ with $W_n = e^{\frac{2\pi i}{n}}$. We consider the Gabor system $\mathcal{G}(g, a, b) = \left\{ M_{bl} T_{ak} g : k = 0, \ldots, \tilde{a} - 1; l = 0, \ldots, \tilde{b} - 1 \right\}$, where the parameters $a$ and $b$ are factors of $n$, i.e. $\tilde{a} = \frac{n}{a}$ and $\tilde{b} = \frac{n}{b}$ are integers.

In the discrete, finite-dimensional case, the Gabor frame operator has a very special structure, the matrix $S$ is zero except in every $\tilde{b}$-th side-diagonals and these side-diagonals are periodic with period $a$. This property can be directly seen by using the *Walnut representation* [9] of the Gabor frame matrix $S = (s_{p,q})$:

**Theorem 2**

$$s_{p,q} = \begin{cases} \tilde{b} \sum_{k=0}^{\tilde{a}-1} \overline{g}_{p-ak} \cdot g_{q-ak} & \textit{for } p - q \equiv 0 \textit{ mod } \tilde{b} \\ 0 & \textit{otherwise} \end{cases}$$

This means $S$ can be represented as a special block matrix, both as a block circulant matrix and as matrix with diagonal blocks [2], which we will call a *Gabor-type* matrix. So the $n \times n$ matrix can be described uniquely by a $b \times a$ matrix $B = (b_{i,j})$ with $b_{i,j} = s_{i \cdot \tilde{b} + j, j}$. This is called the *'non-zero' block matrix* [10]. With this smaller matrix matrix-vector and matrix-matrix multiplication can be calculated very efficiently [3].

## 3. SINGLE PRECONDITIONING OF THE GABOR FRAME OPERATOR

We propose to combine the two following preconditioning methods.

### 3.1. Diagonal Matrices

As a preconditioning matrix the inverse of the diagonal part of the frame operator is used. For every square matrix $A$ let $D(A) = (d_{i,j})$ be the matrix with entries

$$d_{i,j} = \begin{cases} a_{i,i} & i = j \\ 0 & otherwise \end{cases},$$

called the *diagonal part of $A$*.

$$\boxed{P = D(S)^{-1}}$$

**Fig. 1**. The diagonal preconditioning matrix

The diagonal part of a Gabor-type matrix is clearly block-circulant, and therefore also of Gabor-type. This allows us to use the efficient block-matrix algorithms from [10]. If the window $g$ is compactly supported on an interval with length smaller than $\tilde{b}$ then $S$ is a diagonal matrix, see [5]. In this case the inverse matrix is very easy to calculate, by just taking the reciprocal value of the diagonal entries, which are always non-zero for a Gabor frame matrix [3]. Even in the case where the window $g$ is not compactly supported, but $S$ is strictly diagonal dominant, then $S^{-1}$ is well approximated by $D^{-1}$. It is known [11] that in this case the *Jacobi algorithm*, $x_m = D^{-1}(D-S)\,x_{m-1} + D^{-1}g$, converges for every starting vector $x_0$ to $S^{-1}g$. As can be seen from the above formula the Jacobi algorithm is equivalent to preconditioning with $D(S)^{-1}$.

### 3.2. Circulant Matrices

Instead of diagonal matrices circulant matrices can be considered. For a square matrix $S \in M_{n,n}$ let $C(S) = (c_{i,j})_{i,j}$ with $c_{i,j} = \frac{1}{n}\sum_{k=0}^{n-1} s_{k+(j-i),k}$.

$$P = C(S)^{-1}$$

**Fig. 2**. The circulant preconditioning matrix

The two classes of matrices we have investigated so far are connected as follows [2]:

**Theorem 3** *For a circulant matrix $M$ the matrix $\hat{M}$ is diagonal and vice versa.*

Due to properties of the Matrix Fourier Transform [2] the inverse of $C(S)$ can be calculated by using

$$C(S)^{-1} = F_n^* \cdot \left[D\left(F_n \cdot S \cdot F_n^*\right)\right]^{-1} \cdot F_n$$

Therefore the computation of $C(S)^{-1}$ can be done in a very efficient way by using the FFT-algorithm. Analogue to Section 3.1 this can be used as preconditioning matrix.

## 4. DOUBLE PRECONDITIONING OF THE GABOR FRAME OPERATOR

The main result of this work is the double-preconditioning method. In a rather natural way, we will combine the two single preconditioning methods introduced above as seen in Figure 3.

$$P = C\left(D\left(S\right)^{-1} \cdot S\right)^{-1} D(S)^{-1}$$

**Fig. 3**. The double preconditioning matrix

For a basic description of the algorithm see figure 4. In this figure the subscript *'block'* indicates a calculation on the block matrix level, which makes this algorithm very efficient [3]. The expressions $diag_{block}(M)$, $circ_{block}(M)$, $inv_{block}(M)$ and $block(g,a,b)$ stand for the calculation of the block matrix of $D(M)$, $C(M)$, $M^{-1}$ and $S$ respectively. The matrix multiplication on block matrix level is signified by $\bullet_{block}$.

- Parameter: $g$, $a$,$b$
- Initialization: $B = block(g,a,b)$
- Preconditioning :

$$P_1 = inv_{block}\left(diag_{block}\left(B\right)\right)$$
$$S_1 = P_1 \bullet_{block} B$$
$$P_2 = inv_{block}\left(circ_{block}\left(S_1\right)\right)$$
$$S_2 = P_2 \bullet_{block} S_1$$

**Fig. 4**. The double preconditioning algorithm

## 5. NUMERICAL RESULTS

We present two interesting examples, that show the efficiency of this algorithm. For more numerical data refer to [4].

### 5.1. The shapes of the approximated duals

In this first example we will use the double preconditioning matrix to get an approximate dual. For that the double preconditioning matrix itself, $P_2$ in Figure 4, is used as an approximation of the inverse Gabor frame operator.

In this experiment, Figure 5, we see

1. that the different single preconditioning steps can capture certain properties of the dual window but fail to do so for others.

2. the double preconditioning lead to a good approximation of the dual.

This experiment was done with signals of length $n = 144$ using a Hamming window of length $win = 24$, $a = 12$ and $b = 9$. We will use the names *diagonal dual*, *circulant dual* and *double dual* for the window we get when we apply the preconditioning matrices to the original window. The diagonal dual is not similar to the canonical dual near the center, but approximates it well farther away, while the circulant dual just has the opposite property. Opposed to these 'single duals' the 'double dual' seems to combine these properties to become similar to the true dual everywhere.

### 5.2. Iteration

Instead of using the preconditioning matrix as approximation of the inverse, we can iterate this scheme using the Neumann algorithm. This is demonstrated in an example with a Gaussian window, $n = 720$, $a = 24$ and $b = 20$. See Figure 6. We look at the preconditioning steps, the frame algorithm with optimal relaxation parameter and a conjugate gradient method.

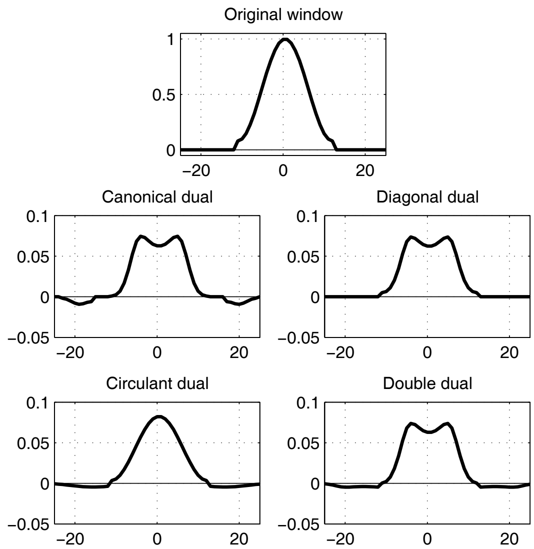


**Fig. 5**. *Windows*: Comparison of original, canonical dual and the approximation of the dual using the preconditioning methods.

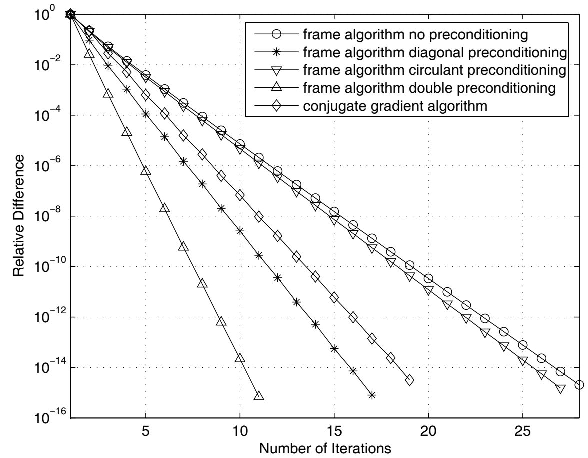


**Fig. 6**. *Convergence with iteration*: Relative difference of iteration steps (Gaussian window, $n = 720$, $a = 24$, $b = 20$.)

In this figure we see that the circulant preconditioning step is only a little bit better than the frame algorithm, iteration-wise. As the time sampling is not very small, this was expected. Diagonal preconditioning is better. The double preconditioning brings a big improvement compared to the single preconditioning methods. It can also be seen that the conjugate gradient algorithm, a method with guaranteed convergence [6], performs worse than double and diagonal preconditioning. Generally our experiments have shown that for increasing $a$ the circulant preconditioning gets worse and for increasing $b$ the diagonal preconditioning gets worse. Double preconditioning is not effected by these deterioration.

## 6. PERSPECTIVES

These algorithms can be very useful in situations, where the calculation of the canonical dual window is very expensive . For example in the situation of *quilted Gabor frames* [12] or the *Time-Frequency Jigsaw Puzzle* [13], globally a frame exists, but it is not a Gabor frame. Therefore there is no dual Gabor window globally, but the dual frame can be approximated by the dual windows of the local Gabor frames. In these cases it might be preferable to use a good and fast approximation of the local Gabor dual windows instead of using a precise calculation of the local canonical dual, as precision will be lost at the approximation of the global dual frame.

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