AN ALGORITHM FOR PARAMETER ESTIMATION OF MULTICOMPONENT CHIRP SIGNALS

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ABSTRACT

This paper presents an algorithm for estimating the parameters of multicomponent chirp signals. The estimator is based on the cubic phase function (CPF), which is efficient to estimate the parameters of monocomponent polynomial phase signals (PPS) with order is less than or equal to 3. When the CPF is dealing with multicomponent chirp signals, the spurious peaks arise and thus the identifiability problem occurs. A new approach based on the transformation called product cubic phase function (PCPF) is proposed to remove this problem. This estimator offers a number of advantages with respect to CPF including improved noise rejection, suppression of cross terms, and elimination of spurious peaks. The algorithm is verified by simulation results.

1. INTRODUCTION

Linear frequency modulated (LFM) signals or chirp signals are often found in signal processing and communication applications such as radar, sonar, biomedicine, seismic analysis, and mobile communications.

Parameter estimation of chirp signal embedded in Gaussian noise has received considerable attention [3-9, 12] for several decades. The time frequency distributions (TFDs), such as the Wigner-Ville distribution (WVD) and its related bilinear class [1-2], are efficient to reveal the instantaneous frequency (IF) over the time-frequency plane and then estimate the parameters using the estimated IF. As to parametric estimation, the maximum likelihood estimation (MLE) [4] plays an important role. The discrete form of the MLE, which is also known as discrete chirps Fourier transform (DCFT), is presented in [5]. To avoid the two-dimensional (2-D) maximization of MLE, Djuric and Kay proposed the use of phase unwrapping and linear regression [3]. It suffers from the ability to analyze the multicomponent signals. The suboptimal techniques including the polynomial phase transform (PPT) [6-7] with order is 2, the Radon-Ambiguity transform (RAT) [8] and the fractional autocorrelation [9], transform the signal into one-dimensional (1-D) parameter space in which only the chirp rate is interested. These techniques have the advantages that they are computationally efficient and can be used for multicomponent signals. Moreover, the fractional Fourier transform (FrFT) [10] also has received considerable attention for its improved noise rejection and ability to resolve closely-spaced chirp signals.

In this paper, we consider the recently proposed transform, called the cubic phase function (CPF) [11] which estimates the instantaneous frequency rate (IFR) first and then used the IFR as initial step to estimate other parameters. In order to improve its performance in the presence of noise and multicomponent signals, a transform called product CPF (PCPF), which exploits the different dependences of the auto terms and the cross terms at the time positions, is proposed next. This is motivated by the similar idea of the product higher-order ambiguity function (PHAF) [12] which discerns the auto terms and the cross terms by utilizing the different dependences on lag.

This paper is organized as follows. In section 2, the problem formulation and the conditions for the spurious peaks are discussed. The PCPF which is supposed to eliminate the spurious peaks along with other benefits is developed in section 3. Section 4 provides the simulation results which validate the proposed estimator. Some discussions and further study are listed in Section 5. Section 6 concludes this paper.

2. PROBLEM FORMULATION

In [11], the CPF was defined as a 2-D bilinear transform of the signal $x(n)$ as

$$CPF(n,\Omega) = \int_0^{+\infty} x(n+\tau)x(n-\tau)e^{-j\Omega\tau^2}\,d\tau\ , \qquad (1)$$

where Ω represents the IFR given by

$$\mathrm{IFR}(n) = d^2\phi(n)/dn^2 \qquad (2)$$

where $\phi(n)$ is the phase of the signal. For the single chirp as

$$s(n) = Ae^{j(a_0+a_1n+a_2n^2)},\ n\in\psi \qquad (3)$$

where $\psi = [-(N-1)/2 : (N-1)/2]$, N is odd, the CPF is derived by directly substituting (3) into (1),

$$\begin{aligned} & CPF(n,\Omega) \\ & = A^2\xi(n)\int_0^{+\infty} e^{j(2a_2-\Omega)\tau^2} d\tau \\ & = \begin{cases} A^2\xi(n)\sqrt{\pi/|8(2a_2-\Omega)|}(1+j), \; 2a_2>\Omega \\ A^2\xi(n)\sqrt{\pi/|8(2a_2-\Omega)|}(1-j), \; 2a_2<\Omega \end{cases} \end{aligned} \quad (4)$$

where $\xi(n)=\exp(j2(a_0+a_1n+a_2n^2))$. It is easy to see that, when $\Omega = 2a_2$, the CPF achieves the maximums along the IFR of chirp signal. The IFR-based algorithm in [11] selects two different time positions to form the equations to estimate the phase parameters. It has been verified both in theory and in simulations that the CPF is highly efficient to analyze the monocomponent signal even at low signal-to-noise ratio (SNR). However, when the CPF is applied to multicomponent signals, the identifiability problem occurs. The analysis of this issue is provided below.

Consider two chirp components:

$$x(n) = A_1e^{j(a_{1,0}+a_{1,1}n+a_{1,2}n^2)} + A_2e^{j(a_{2,0}+a_{2,1}n+a_{2,2}n^2)}. \quad (5)$$

Substituting (5) into (1) yields:

$$\begin{aligned} & CPF(n,\Omega) \\ & = A_1^2e^{j2(a_{1,0}+a_{1,1}n+a_{1,2}n^2)}\int_0^{+\infty} e^{j(2a_{1,2}-\Omega)\tau^2}d\tau \\ & +A_2^2e^{j2(a_{2,0}+a_{2,1}n+a_{2,2}n^2)}\int_0^{+\infty} e^{j(2a_{2,2}-\Omega)\tau^2}d\tau \\ & +A_1A_2z(n)\int_0^{+\infty} e^{j(a_{1,2}+a_{2,2}-\Omega)\tau^2}e^{j\{(a_{1,1}-a_{2,1})+2(a_{1,2}-a_{2,2})n\}\tau}d\tau \\ & +A_1A_2z(n)\int_0^{+\infty} e^{j(a_{1,2}+a_{2,2}-\Omega)\tau^2}e^{j\{(a_{2,1}-a_{1,1})+2(a_{2,2}-a_{1,2})n\}\tau}d\tau \end{aligned} \quad (6)$$

where $z(n)=e^{j\{(a_{1,0}+a_{2,0})+(a_{1,1}+a_{2,1})n+(a_{1,2}+a_{2,2})n^2\}}$. From the above equation, the CPF presents peaks at $\Omega = 2a_{1,2}$ which correspond to the first two terms on the right of equation (6), whereas the cross terms which correspond to the last two terms disperse along the IFR domain. Note that, if

$$(a_{2,1}-a_{1,1})+2(a_{2,2}-a_{1,2})n_c = 0, \quad (7)$$

the result of (6) at $n = n_c$ is

$$\begin{aligned} & CPF(n_c,\Omega) \\ & = A_1^2e^{j2(a_{1,0}+a_{1,1}n_c+a_{1,2}n_c^2)}\int_0^{+\infty} e^{j(2a_{1,2}-\Omega)\tau^2}d\tau \\ & +A_2^2e^{j2(a_{2,0}+a_{2,1}n_c+a_{2,2}n_c^2)}\int_0^{+\infty} e^{j(2a_{2,2}-\Omega)\tau^2}d\tau \\ & +2A_1A_2z(n_c)\int_0^{+\infty} e^{j(a_{1,2}+a_{2,2}-\Omega)\tau^2}d\tau \end{aligned}. \quad (8)$$

The last two terms in (6) merge into one term in (8) which presents a spurious peak at $\Omega = a_{1,2}+a_{2,2}$.

The above result can be generalized to multicomponent case with $K>2$, where K is the number of the components.

(1) The number of cross terms is K^2-K;

(2) The cross terms merge into $(K^2-K)/2$ spurious peaks at the time positions that satisfy the equation as

$$(a_{k,1}-a_{m,1})+2(a_{k,2}-a_{m,2})n = 0, \; \{k,m\}=1,...,K. \quad (9)$$

Hence, the algorithm in [11] is not suitable to estimate the multicomponent chirp signals due to the existence of the spurious peaks. In addition, there is no principle which appropriately selects the time positions that don't present spurious peaks.

3. PRODUCT CUBIC PHASE FUNCTION

The above section establishes the existence of cross terms and the spurious peaks. We also find that the cross terms disperse across the IFR domain and merge spurious peaks when (9) is satisfied, whereas auto terms concentrated on a straight line that located at $\Omega = 2a_{k,2}$, $k=1,...,K$. It can be said that the auto terms are independent of time while the cross terms occur along a linear function of the time position. Hence, it provides the basis for discerning the auto terms from cross terms, even if the cross terms give rise to spurious peaks. In this paper, we define the product CPF (PCPF) as the product of the CPFs at different time positions.

3.1. Definition

Given the set of L different time positions l, the $CPF(l,\Omega)$ corresponding to the set of time is computed by (2) and then the PCPF as the product of the CPFs at different time positions is defined as

$$PCPF(\Omega) = \prod_{l=1}^{N} CPF(l,\Omega). \quad (10)$$

From (10), multiplication of the auto terms in each CPF results in highly distinct peak locates at $\Omega = 2a_{k,2}$. The PCPF provides a number of advantages:

(1) The multiply operation enhances the auto terms due to the fact that auto terms align and weakens cross terms since they disperse in the Ω domain;

(2) The spurious peaks are suppressed or eliminated due to the multiplication of misaligned spurious peaks;

(3) The noise rejection is also improved with respect to CPF.

Obviously, the more sets of time positions used, the better the cross terms suppression capability, but the higher the computational load.

3.2 Estimation Algorithm

In this section, the algorithms for estimating parameters of single and multicomponent chirp signals in additive white Gaussian noise are provided. We make the assumption that the amplitudes of the components are equal or slightly different. For unbalanced signals, the extension can be similarly performed based on the algorithm in [13].

1) **Monocomponent chirp signa**l: Although the PCPF is supposed to suppress the cross terms and spurious peaks of multicomponent signals, the PCPF is also found to

effectively improve the rejection of the additive Gaussian noise. The estimation procedure is as follows.

(1) Estimate the second-order phase coefficient by searching the peak of the PCPF;

(2) Remove the second-order phase contribution by using the demodulation as $x(n) = x(n)\exp(-\mathrm{j}2\pi a_2 n^2)$;

(3) Estimate the first-order phase coefficient using the FFT method or any other subspace method;

(4) Estimate the phase a_0 and the amplitude A by evaluating

$$a_0 = \text{angle}\left[\sum_{n=-(N-1)/2}^{(N-1)/2} x(n)e^{-\mathrm{j}(a_1 n + a_2 n^2)}\right], \tag{11}$$

$$A = \left|\sum_{n=-(N-1)/2}^{(N-1)/2} x(n)e^{-\mathrm{j}(a_1 n + a_2 n^2)}\right| / N. \tag{12}$$

2) **Multicomponent chirp signals**: The algorithm for estimating the multicomponent chirp signals combines the algorithm for estimating the single component signal and the peeling technique [7].

The PCPF may exhibit more than one peak in its spectrum. In this case, we pick the strongest peak and estimate the phase parameter of this component. Then the estimating procedure for single component is implemented to estimate other phase parameters and amplitude of this component. We follow the peeling approach and remove the estimated component from observation data. This completes one recursion of the proposed algorithm. A new recursion is initiated whenever a component is removed from the data. The recursion runs until the spectrum of PCPF does not exhibit strong peaks.

3.3 Computational Cost and Time Position Selection

In general, the PCPF requires about L times more computations than the CPF. However, the computation of CPF can be reduced to the order of $N\log_2 N$ with the use of subband decomposition techniques [11], which is equal to the computation of an N-points fast Fourier transform (FFT). Moreover, L is always a small number, which means the additional cost is not excessive.

Theoretically speaking, there is no limitation on the selection of time positions when the PCPF is applied to multicomponent chirp signals. However, in order to produce better performance, two limitations are preferred:

1) Select the time positions at the middle of the observation time, since the CPFs at time positions close to the beginning and end of observe time are always not distinct. This phenomenon can be also found in WVD and ambiguity function (AF).

2) Adequately space any two time positions in PCPF to sufficiently misalign the cross terms and spurious peaks

4. NUMERICAL ANALYSIS

In this section, we demonstrate the performance of the proposed algorithm by estimating the mean-square error (MSE) through 100 times Monte-Carlo runs in each SNR. Since the estimation algorithm is iterative, it inevitably suffers from error propagation effect. Therefore, the SNR threshold is essentially determined by the threshold related to the correct estimate of second-order parameter. Due to this reason, the performance is demonstrated by the variance of the second-order coefficient.

Example 1 The noiseless signal is generated by (3) and the parameters are chosen to be $A=1$, $a_0=1$, $a_1=\pi/5$, $a_2=\pi/5N$ and $N=515$, and the sampling rate is 1. The SNR varies from -10dB to 5dB in steps of 1dB. The MSEs of $\hat{a}_2$ corresponding to different sets of time are evaluated and presented in Fig. 1. The time sets of L=6 are [-100, -50, 0, 50, 100, 150], whereas the time positions of L=1 and $L=3$ are 0 and [-50, 0, 50], respectively.

The effect of increasing the sets of time is the decrease of the SNR threshold. For L=6, the SNR threshold is about -6dB, which is 3dB decrease of SNR threshold with respect to CPF (L=1). Compared with other chirp rate estimator, the proposed estimator has better performance at low SNRs, i.e., below 0dB and above -6dB.

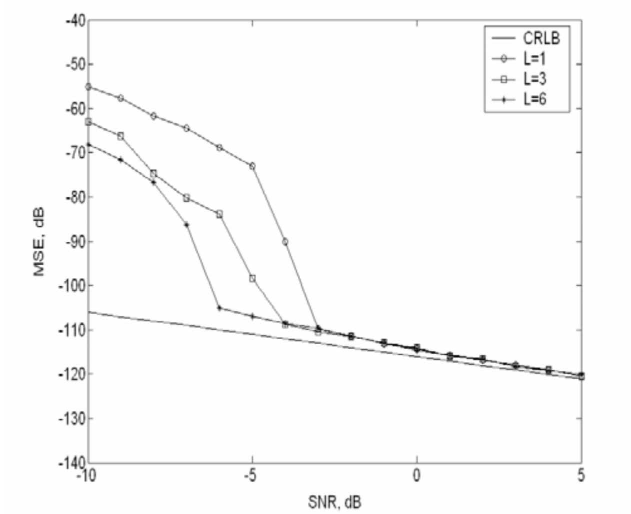

Fig.1. MSE of $\hat{a}_2$ of monocomponent signal versus SNR

Example 2 In this example, we present the performance of the estimator when multicomponent signals are encountered. Two chirp signals are generated. One is the same with the signal in Example 1 and the other is generated by (3) and the parameters are chosen to be $A=1$, $a_{2,0}=1$, $a_{2,1}=2\pi/5$, $a_{2,2}=-2\pi/5N$. For L=3, 6, the selected time positions are the same with Example 1. For L=1, the selected time positions is $n_c=86$ from (7). From this plot, we can observe that the CPF can not estimate the parameter as the SNR increases due to existence of the spurious peaks. However, the PCPF (L=3, 6) removes the identifiability problem and gives small variance at low SNR.

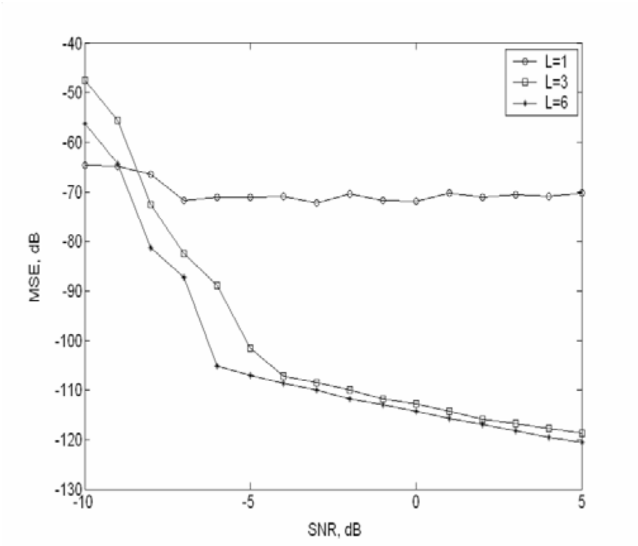

Fig.2. MSE of $\hat{a}_2$ of multicomponent signals versus SNR

5. DISCUSSION

The spurious peaks occur at the time positions subject to a equation like (9) when higher-order chirps are processed by CPF. The PCPF, however, is not able to solve the arising spurious peaks by direct implementation, since both of auto terms and cross terms occur along a function of the time. In particular, the IFR of a cubic chirp, $s(n) = Ae^{j(a_0 + a_1 n + a_2 n^2 + a_3 n^3)}$, is $\Omega = 2a_2 + 6a_3 n$, so we need to remove the influence of a_2 to discern the auto and cross terms. An approach to eliminate coefficient a_2 is to estimate the IFR in the time $n = 0$. Then the PCPF along with IFR scaling as the counterpart of frequency scaling in PHAF follows to suppress the cross terms and spurious peaks. This method is developing.

6. CONCLUSION

In this paper, the spurious peaks are first described when the CPF deals with multicomponent chirp signals. An improved method based on CPF is proposed for removing the identifiability problem. This method is to multiply the CPF at different time positions by exploiting different time dependence of auto terms and cross terms. The PCPF is also effective to improve the rejection of noise and cross terms with respect to CPF. The performance of this method has been evaluated using Monte-Carlo experiments. The results show that the PCPF has better performances than the CPF in the case of both monocomponent and multicomponent signals. The computation cost and time position selection are also discussed.

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