MULTI-RESOLUTION RECONSTRUCTION OF IRREGULARLY SAMPLED SIGNALS WITH COMPACTLY SUPPORTED RADIAL BASIS FUNCTIONS

Arnaud Gelas and Rémy Prost

CREATIS, INSA, UCBL
CNRS UMR 5515, Inserm U630, Lyon, France
email:arnaud.gelas@creatis.insa-lyon.fr, remy.prost@creatis.insa-lyon.fr

ABSTRACT

We propose a novel method for reconstructing d dimensional signals with irregular samples, without any restriction on their positions. We develop a *multi-resolution approximation* scheme using *Compactly Supported Radial Basis Functions* (CSRBFs). Samples are first clustered using principal component analysis and their centroids define CSRBF centers. The mean square error is minimized by selecting centers where the largest local error at the previous level is. We shall prove the effectiveness of our algorithm in one-and two-dimensional cases with Gaussian noise.

1. INTRODUCTION

In many applications, signals can only be sampled at non-uniformly spaced points; for example, due to data loss in communication channels, in image contour analysis, or in data acquisition process, such as surface scanning. For these reasons, the problem of continuous signal reconstruction from irregular samples has received considerable attention. Most previous works make some restrictions on the distribution of points, such as the maximum gap between samples, or ratios between *separation distance* and *fill distance* (see definition in Section 2). However, for many applications we can not consider such restrictions.

When there is no restriction on the distribution of samples, the problem is ill-posed. The most common way to solve this problem is to consider a variational approach (see Section 2) where general solutions are *Radial Basis Functions* (RBFs). Unfortunately, most known RBFs are globally supported, thus the corresponding matrix in the linear system to be solved is dense. In such a case, due to the complexity to solve the linear system, we can only consider a low amount of samples.

Arigovindan et al.[1] consider only a thin plate smoothness energy term, and discretize the solution with non-uniform B-splines. CSRBFs have been introduced[2] to reduce the complexity, in the general variational case. The corresponding matrix in the linear system becomes sparse. However, in the case of highly irregular samples, higher supports are required, and denser the matrix is. Multi-resolution methods [3, 4] have been introduced to overcome this problem. These schemes require the construction of a *center hierarchy* and the computation of *coefficients*. Previous authors use a decimation of samples to define the center hierarchy, and directly interpolate the corresponding sample [3]. The performance of such methods highly depends on the quality of this hierarchy.

In this paper, we will propose a clustering based hierarchy, rather than tuning the decimation process. Our clustering method of samples is based on a principal component analysis, where a cluster's centroid will be used as CSRBF centers. Due to the noise in the data, we will consider the approximation problem instead of interpolation. In our method, we focus on adding centers only where it is required; i.e., where the local error is the largest.

The paper is organized as follows: mathematical preliminaries on variational approach and single scale reconstruction with RBFs are presented in Section 2. In Section 3, we will present a unified multi-resolution formulation, our clustering method and our algorithm. Experiments with 1D and 2D signals prove the effectiveness of our proposal in Section 4 after which a conclusion follows.

2. SINGLE RESOLUTION RECONSTRUCTION WITH RADIAL BASIS FUNCTIONS

Given N irregular samples $\{s_i\}_{i=0,\dots,N-1}$ at locations $\mathcal{P}$ $\{\boldsymbol{p}_i : \boldsymbol{p}_i \in \Omega \subset \mathbb{R}^d\}$, the approximation problem where $f(\boldsymbol{p}_i) \approx s_i$, opposed to the interpolation problem is to construct a continuous function $f(x)$, such that $f(\boldsymbol{p}_i) = s_i$. Since these problems are clearly ill-posed, some a priori knowledge about the function to be reconstructed is required.

The solution of these ill-posed problem can be obtained, from the regularization theory, by variational principle containing data closeness and smoothness information. Solving the reconstruction problem (approximation, or interpolation) consists on finding the function f which minimizes the following general functional:

$$H[f] = \sum_{i=0}^{N-1} \left(f(\boldsymbol{p}_i) - s_i\right)^2 + \lambda \ \left(R[f] - r_0\right) \qquad (1)$$

, where λ is a Lagrange Multiplier, and r_0 is related to the wanted smoothness.

We will say that a function is smoother than another one, if it has less energy in high frequencies domain. The high frequency content of a function can be measured by first high-pass filtering the function, then measuring the power. This suggests the use of the following smoothness term :

$$R[f] = \int_{\mathbb{R}^d} \frac{|\hat{f}(\boldsymbol{\omega})|^2}{\hat{\phi}(\boldsymbol{\omega})} d\boldsymbol{\omega}$$

, where $\hat{}$ indicates Fourier transform in d dimension, $\hat{\phi}$ is some positive function that tends to 0, when $\|\boldsymbol{\omega}\| \to \infty$. If we consider that there is no privileged direction, the smoothness term becomes rotationally invariant, and it can be shown [5] that the function which minimizes the functional (1) has the form:

$$f(\boldsymbol{p}) = \sum_{i=0}^{N-1} \alpha_i \, \boldsymbol{\phi}\left(\|\boldsymbol{p} - \boldsymbol{p}_i\|\right) + \sum_{k=0}^{Q-1} \pi_k \, g_k(\boldsymbol{p}) \qquad (2)$$

, where $\boldsymbol{\phi}$ is a *radial basis functions*, $\{g_k\}_{k=0,\ldots,Q-1}$ is a basis in the m-dimensional null space containing all real-valued polynomials in d variables and of order at most m, $Q = \begin{pmatrix} m-1+d \\ d \end{pmatrix}$, and we require $N \geq Q$; the coefficients $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0,\ldots,N-1}$ and $\pi = \{\pi_k\}_{k=0,\ldots,Q-1}$ satisfy the following linear system:

$$\begin{bmatrix} A_{\boldsymbol{\phi},\boldsymbol{p}} + \lambda I & G^t \\ G & 0 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\pi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{0} \end{bmatrix} \qquad (3)$$

, where $A_{\boldsymbol{\phi},\boldsymbol{p}}(i,j) = \boldsymbol{\phi}(\|\boldsymbol{p}_i - \boldsymbol{p}_j\|)$ and $G(k,i) = g_k(p_i)$. To solve this problem, the only required assumption is that the $A_{\boldsymbol{\phi},\boldsymbol{p}}$ matrix should be conditionally positive definite. However if $A_{\boldsymbol{\phi},\boldsymbol{p}}$ is positive definite no polynomial forms are required.

Eq. (2) is generally referred to *Radial Basis Functions* (RBFs), $\boldsymbol{p}_i$ are traditionally called *centers*, since basis functions are symmetric around them, and α_i *coefficients*.

Despite of the fact that globally supported RBFs are elegant solutions for solving variational problems, in practice they are difficult to use: the matrix $A_{\boldsymbol{\phi},\boldsymbol{p}}$ is dense, though the complexity to solve the linear system (see equation 3) is $O(N^3)$. Wendland[6] makes an excellent survey on the computational aspect about the use of *Radial Basis Functions*, and gives some techniques to decrease the complexity.
Recently some authors have introduced new basis which overcome lots of RBF limitations: *Compactly Supported Radial Basis Functions* [2] (CSRBFs). However, a serious difficulty arises with using this kind of functions. If the support size is too small, the reconstructed signal will not be continuous; on the other hand, the matrix $A_{\boldsymbol{\phi},\boldsymbol{p}}$ becomes dense if the support size is too large. That is why CSRBFs are mainly used for quasi-uniform samples. A point set $\mathcal{P}$ is called *quasi-uniformly distributed* on a bounded domain $\Omega \subset \mathbb{R}^d$, when the quotient of the *fill distance* $h_{\mathcal{P},\Omega}$ ($h_{\mathcal{P},\Omega} = \max_{\boldsymbol{p}\in\Omega} \min_{\boldsymbol{p}_i\in\mathcal{P}} \|\boldsymbol{p} - \boldsymbol{p}_i\|$) and the *separation distance* $q_{\mathcal{P}}$ ($q_{\mathcal{P}} = \min_{\boldsymbol{p}_i \neq \boldsymbol{p}_j \in \mathcal{P}} \|\boldsymbol{p}_i - \boldsymbol{p}_j\|$) is bounded above by a constant.

3. MULTI-SCALE RECONSTRUCTION WITH RADIAL BASIS FUNCTIONS

3.1. Unified Multi-Resolution Scheme

With single scale methods, it is difficult to efficiently reconstruct a signal from irregular samples, and especially with CSRBFs. Multi-resolution recursive schemes [3, 4] have been introduced to overcome this difficulty. Differences between these approaches reside in the way to determine *centers* $\boldsymbol{c}_i^l$ and *coefficients* α_i^l.

Here, all these approaches are unified in the following equations

$$\begin{aligned} f^0(\boldsymbol{p}) &= \sum_{i=0}^{M^0-1} \alpha_i^0 \, \boldsymbol{\phi}\left(\tfrac{\|\boldsymbol{p}-\boldsymbol{c}_i^0\|}{\sigma^0}\right) \\ f^l(\boldsymbol{p}) &= f^{l-1}(\boldsymbol{p}) + d^l(\boldsymbol{p}) \\ d^l(\boldsymbol{p}) &= \sum_{i=0}^{M^l-1} \alpha_i^l \, \boldsymbol{\phi}\left(\tfrac{\|\boldsymbol{p}-\boldsymbol{c}_i^l\|}{\sigma^l}\right) \end{aligned} \qquad (4)$$

, where f^l and d^l are, the approximation or interpolation at the resolution level l, and *details* added to the approximation at the resolution level $l-1$, respectively.

3.2. Multi-Resolution Interpolation

In the interpolation approach [3], data is decomposed into a hierarchy of nested subsets $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \ldots \subseteq \mathcal{P}_k = \mathcal{P}$. The data hierarchy is computed by point removal algorithms with different strategies. Then associated data to centers are interpolated at each level.

Recently, Iske et al. [4] presented a non nested hierarchical data decomposition based on generalized quadtrees. At first, they constructed an initialization by using center of the leaves as *centers* and interpolated the mean value of the data contained in the leaf. However, in some pathological cases, this hierarchy and interpolation would not be efficient.

3.3. Our Proposal: Multi-Resolution Approximation

We will propose a new approach using hierarchical clustering of samples for CSRBF center hierarchy definition (Section 3.3.1). Then we will propose to compute a least square approximation of the samples (Section 3.3.2).

3.3.1. Clustering Method

Here, we will use a binary space partition, based on principal component analysis, for clustering data. From a given cluster

$\mathcal{K}_k$, we will compute the centroid $\boldsymbol{c}_k$ and its associated covariance matrix Cov_k. Then a hyperplane, which splits the cluster into two, is defined by the centroid $\boldsymbol{c}_k$ and the eigenvectors $\boldsymbol{v}_0$ associated to the largest eigenvalue μ_0 of the covariance matrix. By considering of at its lowest level, the whole point set $\mathcal{P}$ as a cluster, we can compute a tree, where the leaves would be samples. It is possible to define various strategies to split one cluster, but in the initialization phase we will only consider the number of samples contained in the cluster $\mathcal{K}_k$, i.e. if $|\mathcal{K}_k| > m$, then $\mathcal{K}_k$ is split, where m is a predefined value.

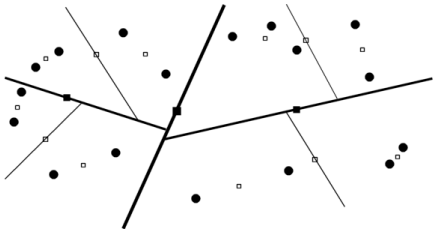


Fig. 1. The Binary Space Partition Clustering Method in the 2D case. The circles represent samples, with lines the splitting hyperplan, and square cluster centroids.

3.3.2. Approximation

Consider the level l, with a list of clusters $\left\{\mathcal{K}_j^l\right\}_{j=\{0,\ldots,M^l-1\}}$ and their respective centroid $\left\{\boldsymbol{c}_j^l\right\}_{j=\{0,\ldots,M^l-1\}}$, and the approximation at the previous level f^{l-1}. We use Wendland's CSRBFs[2] and compute the approximation f^l by minimizing the Mean Square Error MSE^l:

$$MSE^l = \sum_{i=0}^{N-1} \left(s_i - f^l(\boldsymbol{p}_i)\right)^2$$

Finally, it is easy to calculate the coefficients $\boldsymbol{\alpha}^l = \left\{\alpha_j^l\right\}$, where $j = \{0, ..., M^l - 1\}$:

$$\boldsymbol{\alpha}^l = \left(B_{\boldsymbol{\phi}^l} \cdot B^t_{\boldsymbol{\phi}^l}\right)^{-1} \cdot B^t_{\boldsymbol{\phi}^l} \cdot \boldsymbol{r}^l$$

, where $B_{\boldsymbol{\phi}^l}(i,j) = \boldsymbol{\phi}_l\left(\|\boldsymbol{p}_i - \boldsymbol{c}_j^l\|\right)$, $\boldsymbol{\phi}_l(r) = \boldsymbol{\phi}(r/\sigma^l)$, and $\boldsymbol{r}^l(i) = s_i - f^{l-1}(\boldsymbol{p}_i)$.

3.3.3. Algorithm

At first, we shall compute a list of clusters $\mathcal{K}_j^0$, so that they contain at least fewer than m samples, and we compute the first approximation f^0. Then iteratively for a given level l, for all clusters $\mathcal{K}_j^l$ we will compute the local error η_j^l, and compare it with a user defined threshold ϵ.

$$\eta_j^l = \max_{\boldsymbol{p}_i \in \mathcal{K}_j^l} \left|s_i - f^l(\boldsymbol{p}_i)\right| \tag{5}$$

If the error η_j^l is higher than ϵ, the cluster $\mathcal{K}_j^l$ is split into two, and at the next level two new centers will be added. Then we compute a global least square approximation as described above. Note that we will add functions, only where the local error is the largest.

4. RESULTS

We have implemented the proposed algorithm for signals of dimensions $d = 1, 2, 3$. However, we shall only present here some results for $d = 1, 2$.

4.1. 1D Signal

In order to illustrate our approach, we shall consider the signal with $N = 100$ random samples uniformly distributed on $[0, 1]$ as follows:

$$\begin{aligned} s(x) \quad &= 2\ \mathrm{sinc}(10x) + 0.75\ e^{-2.5|x-0.75|}\ \sin(50x + 0.1) \\ &\quad -2\ e^{-10x}\ \cos(55x),\ x \end{aligned} \tag{6}$$

Fig. 2(a) shows the initial approximation with a cluster containing less than m points ($|\mathcal{K}_j^0| < m = 40$). Then other figures present result at different resolutions, $l = 2, 3, 4$, for a given threshold $\epsilon = 0.1$. Fig. 3 illustrates the influence of the user defined threshold ϵ with noisy data. As expected, if ϵ is too low (3(a)), the reconstructed signal captures noise, and if it is too large (3(c)), the reconstructed signal loses the information from the original signal. In Fig. 4, we will present the error according to the threshold value ϵ, for different SNRs. Clearly there is always an optimal value ϵ where the error is the lowest.

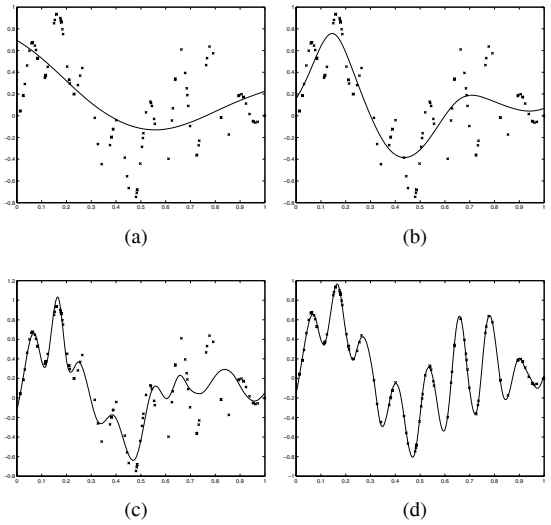
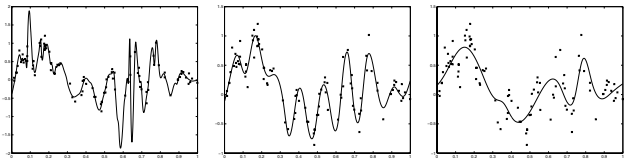


Fig. 2. Approximation of irregularly sampled signal at different resolutions : the lowest resolution 1 (2(a)), resolution 2 (2(b)), resolution 3 (2(c)), and the highest resolution 4 (2(d))



(a) (b) (c)

Fig. 3. Influence of the threshold ϵ on the reconstructed signal, with noisy data (SNR=30dB). With too low $\epsilon = 0.2$ (3(a)), optimal $\epsilon = 0.37$ (3(b)), for too high $\epsilon = 0.7$ (3(c)).

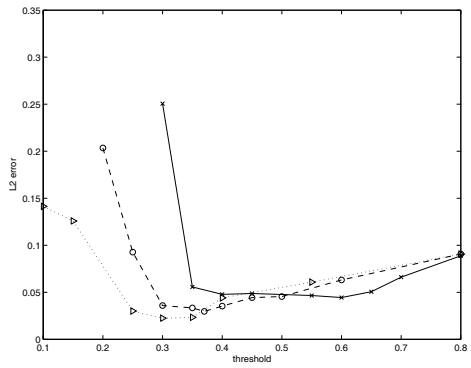


Fig. 4. l_2 error according to the threshold ϵ, for different SNR:27dB(x), 30dB(o), 33dB(▷)

4.2. 2D Signal

We have tested our algorithm on the **peaks** function of Matlab® with $N = 500$ random samples uniformly distributed (figure 5(a)). Figure 5 illustrates the initial approximation with a cluster containing fewer than $m = 100$ samples, and resolutions : 2 and 6.

5. CONCLUSION

We have proposed a new multi-resolution scheme for d dimensional signals approximation with irregular samples without any restriction on their positions. Experimental results prove the effectiveness of this approach for processing noisy data by using the appropriate threshold for CSRBF center selection. In future works, we will propose an estimation of the threshold according to the SNR.

6. REFERENCES

[1] M. Arigovindan, M. Sühling, P. Hunziker, and M. Unser, "Variational image reconstruction from arbitrarily spaced samples: A fast multiresolution spline solution," *IEEE Transactions on Image Processing*, vol. 14, no. 4, pp. 450–460, April 2005.

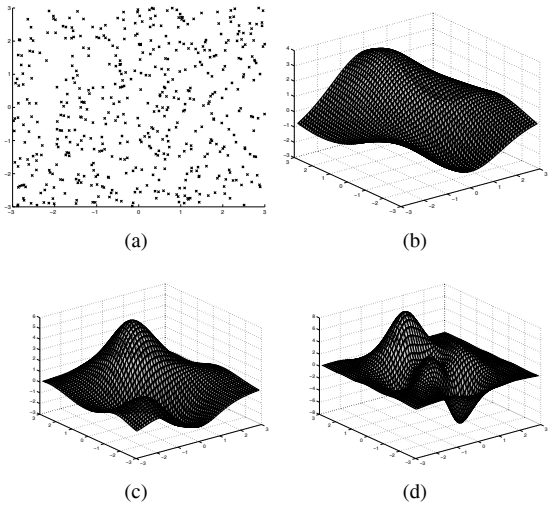
[2] H. Wendland, "Piecewise polynomial, positive definite and compactly supported radial basis functions of minimal degree," *Advances in Computational Mathematics*, vol. 4, pp. 389–396, 1995.

[3] M. Floater and A. Iske, "Multistep scattered data interpolation using compactly supported radial basis functions," *Journal of Computational and Applied Mathematics*, vol. 73, no. 5, pp. 65–780, 1996.

[4] A. Iske and J. Levesley, "Multilevel scattered data approximation by adaptive domain decomposition," *Numerical Algorithms*, vol. 39, no. 1-3, pp. 187–198, july 2005.

[5] F. Girosi, M. Jones, and T. Poggio, "Priors stabilizers and basis functions: From regularization to radial, tensor and additive splines," Tech. Rep., Massachusetts Institute of Technology, Cambridge, MA, USA, 1993.

[6] H. Wendland, "Computational aspects of radial basis function approximation," in *Topics in Multivariate Approximation and Interpolation*, in K. Jetter et al., Ed., pp. 231–256. Elsevier B.V., 2005.



(a) (b) (c) (d)

Fig. 5. Approximation of an irregularly sampled surface. (5(a)) sample locations in a top view. The lowest resolution 1 (5(b)), resolution 2 (5(c)), highest resolution 6 (5(d)).