# PERIODIC NON UNIFORM SAMPLING OF NON BANDLIMITED SIGNALS

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**ABSTRACT**

We consider the periodic non uniform sampling of a class of continuous time non bandlimited signals. Unlike previous work, the periodic non uniform sequence is obtained by retaining a select group of samples from a larger set generated by oversampling the continuous time signal. The oversampled sequence is modeled as the output of a discrete time multirate interpolation filter. Using this model, we propose a number of ways to retrieve the signal and derive necessary and sufficient conditions for signal reconstruction with *FIR digital filters*. The use of FIR filtering insures the stability of the reconstruction process.

## 1. INTRODUCTION

Consider the class of continuous time signals that can be mathematically modeled as

$$x_c(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t-k) \tag{1}$$

where $\phi(t)$ is a *known* function. If $\phi(t)$ has the zero crossing property (also known as the Nyquist(1) property), i.e., $\phi(n) = \delta(n)$ where $\delta(n)$ is the unit sample (impulse) sequence, then, $x(n)$, the uniform periodic samples of $x_c(t)$, is equal to $c(n)$. A classical example of a Nyquist(1) function $\phi(t)$ is $\sin \pi t/\pi t$. In this case, $x_c(t)$ is bandlimited and (1) is the familiar Shannon reconstruction formula. In recent years, a number of researchers have proposed the extension of this idea to the class of *non-bandlimited* signals $x_c(t)$ that can be represented as in (1) with $\phi(t)$ having compact support (For a detailed exposition on the subject, see [1, 2]). Typical examples of compactly supported $\phi(t)$ are the scaling function used in the Wavelet transform and the $N^{th}$ order B-spline function $\beta^N(t)$ [1]. A key result that has emerged from this body of work is that $x_c(t)$ can be reconstructed from the samples $x(n) \triangleq x_c(n)$ *even though* $x_c(t)$ *is not bandlimited* or more specifically, $\phi(n)$ *is not Nyquist*(1). For example, when $\phi(t)$ is $\beta^N(t)$, the zero crossing property is not satisfied and $x_c(t)$ is not bandlimited since it is an $N^{th}$ order spline [3]. However, it is still possible in this case to reconstruct $x_c(t)$ from $x(n)$. To see this, observe that the sampled signal $x(n) = \sum_{k=-\infty}^{\infty} c(k)\phi(n-k)$, i.e., $x(n)$ is the convolution of $c(n)$ with $\phi(n)$. We can therefore recover $c(n)$ from $x(n)$ by using the digital filter $1/\Phi(z)$ where $\Phi(z) = \sum_n \phi(n)z^{-n}$. For the case where $\phi(t)$ is $\beta^N(t)$, $\Phi(z)$ is FIR, has zeros both inside and outside the unit circle and is implemented as a non causal IIR filter to ensure the stability of the reconstruction process [4]. Although this works well for finite length signals like images [5], reconstruction solutions with FIR filters are often desirable due to their simplicity. A formulation that uses the samples of both $\beta^N(t)$ and its derivative to reconstruct $x_c(t)$ with FIR filters has been recently proposed in [6]. In this paper, we show that we can perfectly reconstruct $x_c(t)$ with *FIR filters* by processing *periodic non uniform samples* of $x_c(t)$. However, unlike in [2] where these samples are obtained by directly sampling $x_c(t)$ in a non uniform manner, we generate the periodic non uniform samples by retaining $Q$ samples out of every $MQ$ samples of $x(n) \triangleq x_c(n/M)$ for some integers $Q$ and $M$. In this case, the sequence $x(n)$ is modeled as the output of the interpolation filter of Figure 1 where the box labelled $\uparrow M$ denotes an upsampler $M \geq 2$ and $F(z)$ is the FIR transfer function of $f(n) \triangleq \phi(n/M)$. By adopting this approach, we can derive a number of theoretical results based on multirate DSP techniques that retrieve $c(n)$ from a non uniformly decimated version of $x(n)$ by FIR filtering. The problem of recovering $c(n)$ from a non uniformly decimated version of $x(n)$ has been actually studied in [2] and a *restricted* form of sufficient conditions for FIR reconstruction has been reported. However, the problem in its general form is currently open [2] and a complete solution is presented here. In specific, using first two polyphase components of $F(z)$ and two different choices of $Q$, we derive necessary and sufficient conditions to retrieve $c(n)$ from a non uniformly decimated version of $x(n)$ with FIR filters. These results are then extended for the case of multiple polyphase components and analogous theorems are presented. Several examples are also given to illustrate our main findings.

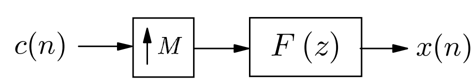


Figure 1: A discrete time multirate model

## 2. PERIODIC NON UNIFORM DECIMATION

An equivalent representation of Figure 1 was derived in [2] and is shown in Figure 2. The box labelled $\downarrow Q$ denotes a downsampler with decimation ratio $Q$. The matrices $\mathbf{R}_j(z)$, $j = 0, 1, \ldots, M-1$ are the $Q$ blocked versions of $R_j(z)$ [7] and $R_j(z)$ are the $M$ polyphase components of $F(z)$, i.e., $F(z) = \sum_{k=0}^{M-1} R_k(z^M)z^k$. Finally, the signals $x_i(n)$ are the $MQ$ polyphase components of $x(n)$, $i = 0, 1, \ldots, MQ-1$. Consider any subset of $Q$ such signals. This subset defines a non uniformly decimated version of $x(n)$. From Figure 2, since $c(n)$ is simply the interleaved version of $c_l(n)$, $l = 0, 1, \ldots, Q-1$, recovering $c(n)$ is achieved if and only if the $Q$ signals $c_l(n)$ can be reconstructed from a $Q$ subset of $x_i(n)$, denoted by $v_l(n)$. Let

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{Q-1}(z) \end{bmatrix} = \mathbf{A}(z) \begin{bmatrix} C_0(z) \\ C_1(z) \\ \vdots \\ C_{Q-1}(z) \end{bmatrix} \quad (2)$$

where the $Q \times Q$ matrix $\mathbf{A}(z)$ is carefully selected. It is clear that a necessary and sufficient condition for retrieving $c(n)$ from periodic non uniform samples of $x_c(t)$ is that $\mathbf{A}(z)$ is non singular. Furthermore, for *an FIR solution*, *det* $\mathbf{A}(z) = \alpha z^{-P}$. Note that the matrix $\mathbf{A}(z)$ depends entirely on $F(z)$, $M$ and $Q$.

## 3. RECONSTRUCTION FROM TWO POLYPHASE COMPONENTS

For simplicity purpose, we assume that $F(z)$ is a causal FIR filter. We emphasize however that this assumption is not necessary for the validity of our analysis. Since $M \geq 2$, we have at least two FIR polyphase components $R_i(z)$ and $R_j(z)$ to work with. Let $R_i(z) = \sum_{k=0}^{N_1} a_k z^{-k}$ and let $R_j(z) = \sum_{k=0}^{N_2} b_k z^{-k}$. Clearly, $N_1 + N_2 \leq N$. Let $Q = N_1 + N_2$. Then, the blocked versions of $\mathbf{R}_i(z)$ and $\mathbf{R}_j(z)$ are given respectively by the following pseudocirculant matrices [7]

$$\mathbf{R}_i(z) = \begin{bmatrix} a_0 & \ldots & a_{N_1} & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & a_{N_1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z^{-1}a_{N_1} & 0 & \ldots & a_0 & \ldots & a_{N_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z^{-1}a_{N_1} & z^{-1}a_{N_1-1} & \ldots & 0 & \ldots & a_0 \end{bmatrix}$$

$$\mathbf{R}_j(z) = \begin{bmatrix} b_0 & \ldots & b_{N_2} & 0 & \ldots & 0 \\ 0 & b_0 & \ldots & b_{N_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z^{-1}b_{N_2} & 0 & \ldots & b_0 & \ldots & b_{N_2-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z^{-1}b_{N_2} & z^{-1}b_{N_2-1} & \ldots & 0 & \ldots & b_0 \end{bmatrix}$$

From the above two matrices, we can now form the $Q \times Q$ matrix $\mathbf{A}(z)$ by selecting the first $N_2$ rows of $\mathbf{R}_i(z)$ and the first $N_1$ rows of $\mathbf{R}_j(z)$ to obtain

$$\mathbf{A}(z) \triangleq \begin{bmatrix} a_0 & a_1 & \ldots & a_{N_1} & 0 & \ldots & & 0 \\ 0 & a_0 & a_1 & \ldots & a_{N_1} & 0 & \ldots & 0 \\ & & \ddots & & & \ddots & & \\ 0 & \ldots & 0 & a_0 & a_1 & \ldots & & a_{N_1} \\ b_0 & b_1 & \ldots & b_{N_2} & 0 & \ldots & & 0 \\ 0 & b_0 & b_1 & \ldots & b_{N_2} & 0 & \ldots & 0 \\ & & \ddots & & & \ddots & & \\ 0 & \ldots & 0 & b_0 & b_1 & \ldots & & b_{N_2} \end{bmatrix}$$

With this particular choice, the matrix $\mathbf{A}(z) \triangleq \mathbf{A_1}$ is a constant matrix, i.e., independent of $z$. If this matrix is non singular, then, its inverse is scalar and the digital reconstruction filters are guaranteed to be FIR. We are now ready to state the first main result of this section.

**Theorem 1.** The matrix $\mathbf{A_1}$ is non singular if and only if the two polynomials $R_i(z)$ and $R_j(z)$ are relatively prime, i.e., do not share a common zero.

Although the above conclusion can be reached by noting that the matrix $\mathbf{A_1}$ is a Sylvester matrix, Theorem 1 can be actually proved using simple linear algebra notions. Unfortunately, due to space limitation, none of the proofs is included in this paper. There is actually a good amount of literature on Sylvester matrices relating, for example, the nullity of $\mathbf{A_1}$ to the number of common zeros (degree of greatest common factor) between the two polynomials (see [8] and the references therein) but we will not pursue these issues here. Instead, we make the following claim.

**The choice of $Q$ is not unique.** Consider now the case where $Q = 2N_{max}$ and $N_{max} = \max(N_1, N_2)$. For example, assuming that $N_2 > N_1$, we can form the following $2N_{max} \times 2N_{max}$ matrix

$$\mathbf{A_2} = \begin{bmatrix} a_0 & a_1 & \ldots & a_{N_1} & 0 & \ldots & & 0 \\ 0 & a_0 & a_1 & \ldots & a_{N_1} & 0 & \ldots & 0 \\ & & \ddots & & & \ddots & & \\ 0 & \ldots & 0 & a_0 & \ldots & a_{N_1} & \ldots & 0 \\ b_0 & b_1 & \ldots & b_{N_2} & 0 & \ldots & & 0 \\ 0 & b_0 & b_1 & \ldots & b_{N_2} & 0 & \ldots & 0 \\ & & \ddots & & & \ddots & & \\ 0 & \ldots & 0 & b_0 & b_1 & \ldots & & b_{N_2} \end{bmatrix}$$

Note that in this case the matrix $\mathbf{A_2}$ is *not* a Sylvester matrix. If $\mathbf{A_2}$ is non singular, then, the reconstruction filters are guaranteed to be FIR. The next theorem provides the corresponding necessary and sufficient conditions.

**Theorem 2.** The matrix $\mathbf{A_2}$ is non singular if and only if the two polynomials $R_i(z)$ and $R_j(z)$ are relatively prime, i.e., do not share a common zero.

## 4. RECONSTRUCTION FROM MULTIPLE POLYPHASE COMPONENTS

Given the previous exposition and assuming that $M$ is *fixed*, is it still possible to reconstruct $x_c(t)$ by means of FIR filters *even if we can not find two coprime polyphase components* ? The need for more than two polyphase components is obvious but how do we choose the parameter $Q$ in this case ? Furthermore, given a particular choice of $Q$, what are the necessary and sufficient conditions for the existence of FIR solutions ? More generally, assuming that there does not exist a set of $P-1$ polyphase components that are relatively prime and, given $Q$, $M$ and, a number of polyphase components $2 < P \leq M$, what are the necessary and sufficient conditions for recovering $c(n)$ by *FIR filters* ? To answer these questions, we start with the following observation.

**Observation.** The requirement that every combination of $P-1$ polyphase components share at least a common zero implies in particular the following relation

$$N_{max} \geq N_k \geq N_{min} \geq P-1 \qquad k = 1, 2, \ldots, P-1 \quad (3)$$

where $N_k$ is the order of the $k^{th}$ polyphase component and $N_{max}$ and $N_{min}$ are the orders of the maximum and minimum polyphase components respectively.

Recall now equation (2). The problem of recovering $c(n)$ perfectly is therefore equivalent to choosing $Q$ and consequently $\mathbf{A}(z)$ such that $det\ \mathbf{A}(z) = \alpha z^{-p}$. One way to satisfy the determinant equality is to choose the rows of $\mathbf{A}(z)$ such that $\mathbf{A}(z) = \mathbf{A}$ (independent of $z$). In this general case, the matrix $\mathbf{A}$ will be a square sub-matrix of a *Generalized* Sylvester matrix [8] and will reduce to the Sylvester matrix $\mathbf{A_1}$ when using only two polyphase components. One possible choice of the parameter $Q$ is given by the next lemma.

**Lemma.** Assuming we use $P \leq M$ polyphase components to find $c(n)$, the parameter $Q$ is then given by

$$Q = \frac{\sum_{k=1}^{P} N_k}{P-1} \tag{4}$$

where $N_k$ is the order of the $k^{th}$ polyphase component.

**Constraints on the Polyphase Components.** Equation (4) puts explicitly some constraints on the orders of the $P$ FIR polyphase components. First, when $N_k = N_{max}$ for all $k$, $N_{max}$ has to be divisible by $P-1$. Second, the condition $Q - N_{max} > 0$ has to be satisfied in order for equation (4) to be valid. This condition in turns implies the following relation between the orders of the polyphase components $R_k(z)$, $k = 1, 2, \ldots, P$

$$N_{max} > \sum_{k=1}^{P} (N_{max} - N_k) \tag{5}$$

We note that in the two polyphase case, these conditions do not exist since $P = 2$ and $Q = N_2 + N_1 > N_k$ for $k = 1, 2$. Finally, from (3) (the other constraint on the orders), we can deduce the following relation

$$Q = \frac{\sum_{k=1}^{P} N_k}{P-1} \geq \frac{P\ (P-1)}{P-1} \qquad \Rightarrow \qquad Q \geq P \tag{6}$$

**Not all choices of $Q$ are good.** A tempting generalization of $Q = N_1 + N_2$ of section 3 is the choice $Q = \sum_{k=1}^{P} N_k$. Furthermore, with this choice, there are basically no constraints on the orders of the polyphase components. Nevertheless, we can show that in this case, the coprimeness of the polyphase components does not guarantee the non singularity of its corresponding matrix.

**Non uniqueness of $Q$.** Similar to the two polyphase case, we can set

$$Q = 2 \max_{k}\ N_k \qquad \forall\ k = 1, 2, \ldots, P \tag{7}$$

Denote $Q$ in (4) by $Q_1$ and $Q$ in (7) by $Q_2$. Then, we have the following relation

$$Q_1 = \frac{\sum_{k=1}^{P-1} N_k}{P-1} + \frac{N_P}{P-1} \leq N_{max} + \frac{N_{max}}{P-1} \leq 2N_{max} = Q_2$$

with equality if and only if $N_k = N_{max}$ for all $k$ **and** $P = 2$, i.e., $N_1 = N_2 = N_{max}$. It follows that

$$Q_2 - Q_1 = \frac{\sum_{k=1}^{P-1} (N_{max} - N_k)}{P-1} + \frac{N_{max}(P-2)}{P-1} \geq 0 \tag{8}$$

**Problem Setup.** Assume you are given $P$ FIR polyphase components $R_k(z)$, $k = 1, 2, \ldots, P$, such that *every* subset of $P-1$ polynomials share at least one common zero. Assume further that these polynomials have order $N_k$ such that (5) and (3) are satisfied and that, without loss of generality, the maximum order $N_{max} = N_P$. Define the *integers* $Q_1$ and $Q_2$ as in (4) and (7) respectively. Then, we can always construct two matrices, namely a $Q_1 \times Q_1$ matrix $\mathbf{A_1}$ and a $Q_2 \times Q_2$ matrix $\mathbf{A_2}$ where $\mathbf{A_1}$ is embedded in $\mathbf{A_2}$ as follows

$$\mathbf{A_2} = \begin{bmatrix} \mathbf{A_1} & \mathbf{0} \\ \mathbf{C} & \mathbf{B} \end{bmatrix} \tag{9}$$

In this case, $\mathbf{0}$ is the $Q_1 \times (Q_2 - Q_1)$ zero matrix, $\mathbf{C}$ is an $(Q_2 - Q_1) \times Q_1$ arbitrary matrix and, $\mathbf{B}$ is a Toeplitz lower triangular $(Q_2 - Q_1) \times (Q_2 - Q_1)$ with the main diagonal element is the *non zero* coefficient of $N_{max}$. Note that $Q_2 - Q_1$ is defined by (8) and for $P = 2$, $Q_2 - Q_1 = N_2 - N_1$. The matrix $\mathbf{A_2}$ is obtained by forming the $Q_2 \times Q_2$ generalized Sylvester matrix as follows

$$\mathbf{A_2} \triangleq \left[\begin{array}{cccccccc}
a_0 & a_1 & \cdots & a_{N_1} & 0 & \cdots & 0 & 0 \\
0 & a_0 & a_1 & \cdots & a_{N_1} & 0 & \cdots & 0 \\
 & & & \ddots & \ddots & & & \\
0 & \cdots & 0 & a_0 & \cdots & \cdots & \cdots & a_{N_1} \\
 & & \vdots & & \vdots & & \vdots & \\
 & & \vdots & & \vdots & & \vdots & \\
b_0 & b_1 & \cdots & b_{N_m} & 0 & \cdots & 0 & 0 \\
0 & b_0 & b_1 & \cdots & b_{N_m} & 0 & \cdots & 0 \\
 & & & \ddots & \ddots & & & \\
0 & \cdots & 0 & b_0 & \cdots & \cdots & \cdots & b_{N_m} \\
 & & \vdots & & \vdots & & \vdots & \\
 & & \vdots & & \vdots & & \vdots & \\
c_0 & c_1 & \cdots & c_{N_P} & 0 & \cdots & 0 & 0 \\
0 & c_0 & c_1 & \cdots & c_{N_P} & 0 & \cdots & 0 \\
 & & & \ddots & \ddots & & & \\
0 & \cdots & 0 & c_0 & \cdots & \cdots & \cdots & c_{N_P}
\end{array}\right]$$

We emphasize that the last Sylvester block has size $N_{max}$ and not $Q - N_{max}$. The construction of $\mathbf{A_2}$ is illustrated in the next example.

**Example.** Assume that $P = 3$ and let

$$\begin{aligned}
R_1(z) &= (1 - z^{-1})(1 - 3z^{-1}) = 1 - 4z^{-1} + 3z^{-2} \\
R_2(z) &= (1 - 2z^{-1})^2(1 - 3z^{-1}) = 1 - 7z^{-1} + 16z^{-2} - 12z^{-3} \\
R_3(z) &= (1 - 2z^{-1})^2(1 - z^{-1}) = 1 - 4z^{-1} + 3z^{-2} - 4z^{-3}
\end{aligned}$$

It follows that $N_{max} = N_3 = 3$, $Q_1 = 4$, $Q_2 = 6$ and $Q_1 - N_{max} = 1 > 0$. We can therefore express the $6 \times 6$ matrix $\mathbf{A_2}$ as follows

$$\mathbf{A_2} \triangleq \left[\begin{array}{cccccc}
1 & -4 & 3 & 0 & 0 & 0 \\
0 & 1 & -4 & 3 & 0 & 0 \\ \hline
1 & -7 & 16 & -12 & 0 & 0 \\ \hline
1 & -5 & 8 & -4 & 0 & 0 \\
0 & 1 & -5 & 8 & -4 & 0 \\
0 & 0 & 1 & -5 & 8 & -4
\end{array}\right]$$

Alternatively, the matrix $\mathbf{A_2}$ can be represented as in (9)

$$\mathbf{A_2} = \left[\begin{array}{cccc|cc} 1 & -4 & 3 & 0 & 0 & 0 \\ 0 & 1 & -4 & 3 & 0 & 0 \\ 1 & -7 & 16 & -12 & 0 & 0 \\ 1 & -5 & 8 & -4 & 0 & 0 \\ \hline 0 & 1 & -5 & 8 & -4 & 0 \\ 0 & 0 & 1 & -5 & 8 & -4 \end{array}\right]$$

**Lemma.** $\mathbf{A_2}$ is non singular if and only if $\mathbf{A_1}$ is non singular.

We can now state the main result of this section.

**Theorem 3.** Given any $P-1$ (out of $M$) polyphase components $R_k(z)$ of order $N_k$ respectively, $k = 1, 2, \ldots, P-1$, with at least one common zero between them and a polyphase component $R_P(z)$ of order $N_{max}$. Let $Q_1 = P$. Then, the $Q_1 \times Q_1$ matrix $\mathbf{A_1}$ is non singular if and only if all the polyphase components $R_k(z)$, $k = 1, 2, \ldots, P$, are relatively prime.

Note that, by Lemma 1, the necessary and sufficient conditions of Theorem 3 also imply the non singularity of the matrix $\mathbf{A_2}$.

**Example.** Assume that $P = 3$ and let

$$\begin{aligned} R_1(z) &= (1+z^{-1})(1+2z^{-1}) = 1+3z^{-1}+2z^{-2} \\ R_2(z) &= (1+z^{-1})(1-z^{-1}) = 1-z^{-2} \\ R_3(z) &= (1+2z^{-1})(1-z^{-1}) = 1+z^{-1}-2z^{-2} \end{aligned}$$

In this case, $N_{max} = 2$, $Q_1 = 3 = P$, $Q_2 = 4$ and $Q_1 - N_{max} = 1 > 0$. The $4 \times 4$ matrix $\mathbf{A_2}$ is given by

$$\mathbf{A_2} = \left[\begin{array}{cccc} 1 & 3 & 2 & 0 \\ \hline 1 & 1 & -2 & 0 \\ \hline 0 & 1 & 1 & -2 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

which can be represented as follows

$$\mathbf{A_2} = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 \\ \hline 1 & 1 & -2 & 0 \end{array}\right]$$

The matrix $\mathbf{A_1}$ is therefore

$$\mathbf{A_1} = \left[\begin{array}{ccc} 1 & 3 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

and $\det(\mathbf{A_1}) = 6$ implying the existence of an FIR reconstruction scheme with 3 polyphase components.

The previous theorem holds only for the special case of $Q_1 = P$. From (6), we know that, in general, $Q_1 \geq P$. Unfortunately, for $Q_1 > P$, the non singularity of the matrix $\mathbf{A_1}$ is not equivalent to the polyphase components being relatively prime. More specifically, $\mathbf{A_1}$ is singular if the polyphase components have a common zero. However, the coprimeness of the polynomials is not sufficient for the non singularity of the matrix $\mathbf{A_1}$ as we show next.

**Example.** Assume that $P = 3$ and let

$$\begin{aligned} R_1(z) &= (1+z^{-2})(1+z^{-1})(1-z^{-1}) = 1-z^{-4} \\ R_2(z) &= (2+z^{-2})(1+z^{-1})(2-z^{-1}) \\ &= 4+2z^{-1}+z^{-3}-z^{-4} \\ R_3(z) &= (3+z^{-2})(1-z^{-1})(2-z^{-1}) \\ &= 6-9z^{-1}+5z^{-2}-3z^{-3}+z^{-4} \end{aligned}$$

In this case, $N_{max} = 4$, $Q_1 = 6 > P = 3$, and $Q_1 - N_{max} = 2 > 0$. The $6 \times 6$ matrix $\mathbf{A_1}$ is given by

$$\mathbf{A_1} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 4 & 2 & 0 & 1 & -1 & 0 \\ 0 & 4 & 2 & 0 & 1 & -1 \\ 6 & -9 & 5 & -3 & 1 & 0 \\ 0 & 6 & -9 & 5 & -3 & 1 \end{array}\right]$$

The matrix $\mathbf{A_1}$ has rank 5 and is therefore singular. However, $R_1(z)$, $R_2(z)$ and $R_3(z)$ do not share a common zero.

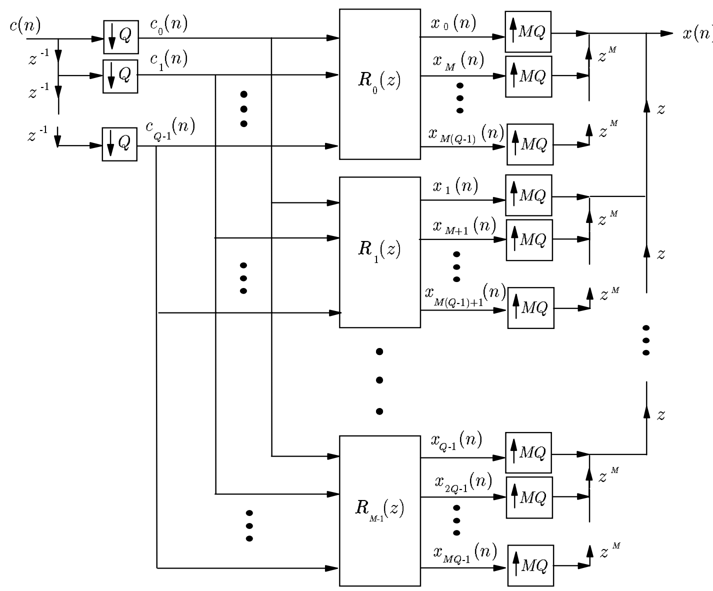


Figure 2: Equivalent representation of Figure 1

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