A Level-Crossing Sampling Scheme for Non-Bandlimited Signals

Karen Guan
ECE, UIUC
Urbana, Illinois, 60801
Email: kguan@uiuc.edu

Andrew C. Singer
ECE, UIUC
Urbana, Illinois, 60801
Email: acsinger@uiuc.edu

Abstract— **We propose a level-crossing A/D (LCA/D) converter which can be modelled with an oversampling A/D followed by a low resolution quantizer. In this paper we will study the reconstruction of non-bandlimited inputs processed by such a system and compare its performance to uniform sampling.**

Index Terms– quantization, level-crossing sampling, Haar wavelets, Cramer-Rao bound.

I. Introduction

In this paper we focus our study on the reconstruction of real non-bandlimited periodic signals in both deterministic and stochastic frameworks. The more conventional uniform sampling, better known as Nyquist sampling, is related to the approximation used in the Riemann sum, shown in Fig.1(a), where amplitudes are taken at fixed intervals. When the signals are bandlimited, Nyquist sampling can reconstruct them perfectly in the $\mathcal{L}^2$ sense, assuming samples are known with infinite precision.

For some other signal types of interest, for example, non-bandlimited waveforms, Nyquist sampling neither sufficiently nor efficiently captures the characteristics of the input. The classical approach is to prefilter the non-bandlimited signals with an anti-aliasing filter at sufficiently high frequency and then sample the resulting bandlimited field. Sometimes this approach cannot be implemented, such as in certain sensor networks where nodes have limited sensing and processing power. Other times it is not desirable, especially when information embedded in higher frequency content is valuable. As such, aliasing error is inevitable. This prompts us to investigate an alternative sampling scheme.

Sampling by level-crossing takes its cue from the Lebesgue integral that approximates with a fixed set of amplitude values, and samples are taken instead on the time axis, as shown in Fig.1(b). This form of approximation follows the characteristics of the waveform. We sample more often when the waveform is rapidly varying or is bursty, and less when otherwise. As such, it lets the signal dictate the frequency of sampling and quantization.

For example, one type of signals that benefits from level-crossing sampling is the non-bandlimited random telegraph waveform. Sampling it uniformly will certainly incur aliasing errors. A better way is to extract the nonuniform transitions using zero-crossing sampling [3],[5], since the information content of the signal lies entirely in these crossings. The signal can thus be perfectly reconstructed. This leads us to believe that sampling by level-crossing has certain advantages that warrant further exploration. In fact, representative works by [1], [2], and [4] have looked into zero/level-crossing representation of various classes of waveforms such as wavelets, and bandlimited inputs. Here we want to illustrate the advantages of sampling by level-crossing over uniform sampling. Specifically, we analyze a class of finite-dimensional, non-bandlimited Haar wavelet series $\mathcal{W}_H$, defined in detail in the following section.

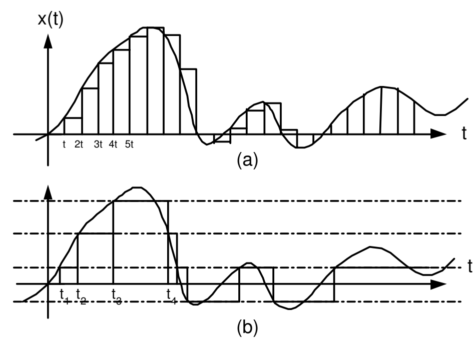

Fig. 1. a) Approximation of a waveform by the Riemann sum, b) by Lebesgue sum.

II. The considered non-bandlimited field

Let $x(t)$ be a real deterministic signal with finite time support on $[0, T]$, $T > 0$. We assume $x(t)$ is a non-bandlimited Haar wavelet series, $x(t) \in \mathcal{W}_H$,

$$\mathcal{W}_H = \{x(t) : x(t) = \sum_{k=0}^{M-1} \sum_{l=0}^{2^k-1} a_{kl} \sqrt{\frac{2^k}{T}}\ \psi(2^k\, t - lT)\}, \quad (1)$$

where $\psi(t)$ is a Haar wavelet:

$$\psi(t) \equiv \begin{cases} 1, & 0 \leq t < \frac{T}{2}; \\ -1, & \frac{T}{2} \leq t < T; \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Each $\psi_{kl}(t) = \sqrt{\frac{2^k}{T}}\ \psi(2^k\, t - lT)$ is a Haar wavelet recoverable by zero-crossing. A sum power constraint is imposed upon $\mathcal{W}_H$ as well:

$$\frac{1}{T}\int_0^T |x(t)|^2 dt \le E. \tag{3}$$

As such, $\mathcal{W}_H$ is a set of non-bandlimited signals with finite energy and finite degrees of freedom, $2^M - 1$ to be exact, on finite time support. We want to reconstruct such waveforms using the level-crossing sampling scheme, then illustrate its merits over the periodic sampling method.

III. Level-crossing A/D model

Let $\mathcal{L} = (l_1, \ldots, l_m, \ldots, l_{|\mathcal{L}|})$ be an ordered set of distinct discrete levels uniformly spaced δ apart, $\delta > 0$. Each l_m requires only finite resolution. We define a level-crossing A/D (LCA/D) as the superposition of a level sampler $\mathcal{L}_\delta$ and the uniform sampling operation $x_n = x(n\tau)$, where τ is the sampling period.

Definition 3.1: The level sampler $\mathcal{L}_\delta$ with a fixed spacing δ is the mapping $\mathcal{L}_\delta$: $\mathbb{R} \to \delta\,\mathbb{Z}$:

$$\mathcal{L}_\delta(x) = \delta \lfloor x/\delta \rfloor.$$

A waveform $x(t)$ is said to have a level-crossing at t if $\mathcal{L}_\delta(x(t^-)) \neq \mathcal{L}_\delta(x(t^+))$.

Letting $\mathbf{t} = (t_1, t_2, \ldots, t_i, \ldots, t_Q) \in [0, L]$ be the resulting set of crossing instants. The level-sampled signal $\mathcal{L}_\delta(x)$ is then uniformly sampled to become a discrete-time, discrete-amplitude (DTDA) signal $\mathcal{L}_\delta(x)_n$.

IV. Computing the MSE

A. MSE and the bit rate in a deterministic framework

In a deterministic environment, the accuracy of reconstruction depends on the resolution of the samples. Here we are motivated by [3] to use the encoding bit rate to evaluate the merit of level-crossing. Specifically, for a fixed fidelity of reconstruction, we will compare the encoding bit rate of the level-crossing samples to that of the uniform samples.

Now consider waveforms in $\mathcal{W}_H$ that have exactly Q transitions, $1 \le Q \le 2^M$, during the signal period T. Their locations are unknown.

Let us characterize the resolution of LCA/D with bit rate B_L. The information of $\mathcal{L}_\delta(x)$ is entirely embedded in its transition samples $(t_i, x(t_i))_{1\le i\le Q}$, where $x(t_i) = l_m$. Each amplitude can be represented with $\log_2 |\mathcal{L}|$ bits, and since transitions for signals of class $\mathcal{W}_\mathcal{H}$ happen on integer multiple of $\frac{T}{2^M}$, each t_i requires M bits resolution. As such,

$$B_L = \frac{(\log_2 |\mathcal{L}| + M)\cdot Q}{T}. \tag{4}$$

In the case of periodic sampling, amplitude samples, K bits each, are stored sequentially, requiring a bit rate of

$$B_N = \frac{K \cdot 2^M}{T}. \tag{5}$$

The sampled signal is generally different from the original and incurs a nonzero error. We make the conventional assumption that the error is uniformly distributed white noise independent of the input. Let the resolution of the LCA/D be given by $\delta = \frac{X_m}{|\mathcal{L}|}$, where X_m be the maximum amplitude range of $x(t)$ imposed by the power constraint in Eq.2. For a fixed mean squared error, we find that level-crossing requires exponentially smaller bit rate than periodic sampling does:

$$B_L = 2^{-(M-\log_2 Q)} B_N + \frac{Q}{T} M. \tag{6}$$

Note that in order to keep the comparison fair, the periodic samples are quantized with $\mathcal{L}$ as well, thus $K = \log_2 |\mathcal{L}|$. This highlights the efficiency of level-crossing sampling, in that it lets the signal dictate the amount of sampling. For example when $M = 10$, and $Q = 2^6$, then the $B_L \approx 2^{-4} B_N$.

B. Q level-crossing samples vs. Q uniform samples in a stochastic framework

In this section, we will highlight the advantage of level-crossing sampling by comparing its Cramer-Rao bound (CRB) to that achieved by uniform sampling. To keep the comparison fair, Q samples are used in each scheme. When the waveform is bursty, we will show that level-crossing has a lower CRB.

Consider the setup shown in Figure 2. The input is transmitted over an AWGN channel, where the receiver uses the output of the channel $y(t)$ to construct estimates $\hat{\mathbf{a}}$. The samples are obtained by a sample-and-hold circuit of finite bandwidth, modelled as:

$$y(t_i) = \frac{1}{\tau'}\int_{t_i}^{t_i+\tau'} y(t) dt. \tag{7}$$

The samples are Gaussian, $y(t_i) \sim \mathcal{N}\left(x(t_i), \frac{N_{\tau'}}{2}\right)$, $N_{\tau'} = \frac{N_o}{\tau'}$.

Since the channel is noisy, distortion is inevitable. The variance of estimates based on Q observations is bounded by the CRB:

$$Var(\hat{a}_{kl}) \ge \frac{N_{\tau'}/2}{\sum_{i=1}^{Q} \psi_{kl}^2(t_i)}. \tag{8}$$

The lower bound can be met in principle, under the Gaussian assumption, with a simple linear unbiased estimator. The denominator is a signal-dependent term that dictates the lower-bound. Specifically, it depends on where samples are taken. Now let us consider the following example, an simplified but representative scenario of UWB signalling, to gain an insight into the advantage of level-crossing sampling.

An example of estimating a bursty waveform: Suppose $x(t)$ is bursty, such that its energy is concentrated in an unknown small interval $(a, b) \in [0, T]$, i.e. $\frac{1}{T}\int_a^b |x(t)|^2 dt = E$, of length $b - a < \frac{T}{Q}$. Furthermore assume that only Q unknowns are nonzero.

Let Q samples, $(t_i, y(t_i))_{i=1}^{Q}$, be obtained by level-crossing during (a, b). Since level-crossing lets the signal dictate when to sample, most samples are located within (a, b), where the waveform resides. Assume the t_i's have perfect precision, then estimates made with these Q samples have variance bounded by:

$$Var(\hat{a}_{kl} \neq 0) \ge \frac{T}{\alpha \cdot Q} \cdot \frac{N_{\tau'}}{2},\ \ 1 \le \alpha \le 2^k. \tag{9}$$

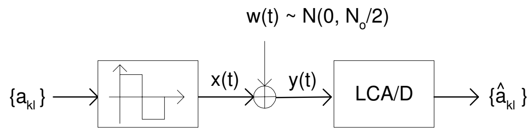

Fig. 2. The stochastic framework considered in this paper.

The other a_{kl}'s are zero anyway, so they need not be estimated. In other words, level-crossing lets the signal decide where samples are to be taken, thus allowing us to obtain only the useful ones.

On the other hand, when the Q samples are obtained uniformly, spaced $\tau = \frac{T}{Q}$ apart, little useful information is obtained. They carry no information on the parameters modulated by bases with time support less than sampling period. The $\psi_{kl}(t)|_{t_i=i\tau}$ take on values $\pm\sqrt{\frac{2^k}{T}}$ and 0, therefore it is entirely possible that for certain a_{kl}, $\sum_{i=1}^{Q} \psi_{kl}^2(t_i)$ sums up to 0. As such, the Fischer's information is 0, and the bound provided by Eq.(8) is invalid. As such, no estimation can be made.

The worst scenario occurs when $(a, b) \subset (i\tau, (i+1)\tau]$, where the signal is entirely missed by sampling. All samples $y(i\tau) = 0, \forall\, i$, thus estimates $\hat{a}_{kl} = 0, \forall\, k, l$. Otherwise only 1 out the Q samples is taken within the interval (a, b) where the signal resides. As such, at most M out of the Q unknowns can be estimated, and the CRB of nonzero estimates is much larger:

$$Var(\hat{a}_{kl} \neq 0) \geq \frac{T}{2^k}\frac{N_{\tau'}}{2}. \tag{10}$$

We have shown that uniform sampling is particularly unsuited for this type of input. It mathematically prove the intuitive fact that piecewise constant functions are more efficiently sampled using a level-crossing scheme.

V. Conclusion

We proposed a new sampling scheme by level-crossing which allows the input signal to dictate how much we sample. We focused our study of LCA/D on a class of finite dimensional, non-bandlimited signals on finite time support. It was shown in both a deterministic and a stochastic framework, for certain types of waveforms, LCA/D outperforms periodic sampling. Furthermore, we can extend our study to other classes of signals, such as truncated Fourier Series, or Haar wavelet series with non-integer dilation and translation parameters.

References

[1] Ian F.Blake and William C.Lindsey, *Level-Crossing Problems for Random Processes,* IEEE Transactions on Information Theory, Vol.IT-19, NO.3, pp.295-313, May, 1973.

[2] B.Logan, "Information in the zero-crossings of band pass signals," Bell Syst. Tech.Journal,Vol.56, pp.510-525, 1977.

[3] A.A.G.Requicha, *The Zeros of Entire Functions,* Proc.of IEEE, vol.68,pp.308-328, 1980.

[4] Stephane Mallat, *Zero-Crossings of a Wavelet Transform,* IEEE Transactions on Information Theory, vol.37, NO.4, pp.1019-1033,July 1991.

[5] Pina Marziliano and Martin Vetterli, *Reconstruction of Irregularly Sampled Discret-Time Bandlimited Signals with Unknown Sampling Locations,* IEEE Tran. Sig. Proc., vol.48, pp.3462-3471, 2000.

[6] Z.Cvetković and Martin Vetterli, *On Simple Oversampled A/D Conversion in $L^2(\mathbb{R})$,* IEEE Tran. Info.Theory, vol.47,pp.146-153, 2001.