# ENUMERATION AND PARAMETRIZATION OF DISTINCT DOWNSAMPLING PATTERNS IN TWO-DIMENSIONAL MULTIRATE SYSTEMS

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**ABSTRACT**

Resampling matrices are essential in multidimensional multirate signal processing. However, different resampling matrices may correspond to the same resampling pattern. It is surely more effective to study such distinct patterns rather than study exhaustively all the possible cases, which is infinite. Motivated by this, we study the resampling patterns for a given decimation ratio in this paper. Our analysis enumerates the number of distinct resampling patterns for any prime decimation ratio, and provides an upper bound on the number for any composite decimation ratio. Parametrization of these distinct patterns is also proposed. Among these patterns, patterns which are nonseparable are also enumerated and parameterized separately.

## 1. INTRODUCTION

Two essential building blocks of a multirate system are the decimator for the analyzed signal and the interpolator for the synthesized signal. For one-dimensional (1-D) signal, uniform resampling is intuitively simple and hence popular. It has been thoroughly studied in many publications such as [1]. However, they become exponentially varied and complicated with the dimension when higher dimensional signal is considered. This born difficulty has attracted quite a few research concentration in recent decades. Many meaningful results have been presented in literature. Some are specifically focused on interchangeability of decimator and interpolator [2],[3],[4],[5]. Some have a more comprehensive scope [6],[7],[8]. According to these results, the nature of resamplers has been well understood.

This paper also falls in this class. The resampling patterns for different resampling matrices are investigated. Approaches to parameterize all distinct resampling patterns are provided. Since upsampling is just the reciprocal counterpart of downsampling, in the following we limit our study to decimation matrices without loss of generality. Similar to most of previous literatures, we only focus on 2-D systems, which is of most practical importance.

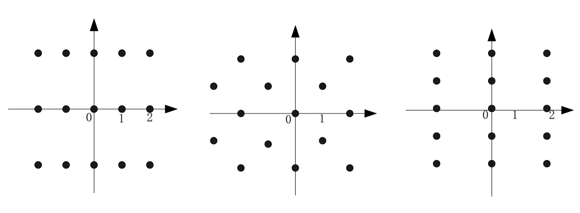

**Fig. 1**. Three possible lattices for $|\mathbf{M}| = 2$

In 2-D systems, the decimator is a $2 \times 2$ non-singular integer matrix, denoted as $\mathbf{M}$. The decimation matrix $\mathbf{M}$ keeps those samples that are on the lattice generated by $\mathbf{M}$. The lattice generated is the set of integer vectors $\mathbf{m}$ such that $\mathbf{m} = \mathbf{M}\mathbf{n}$ for some integer vector $\mathbf{n}$. For example, in Fig. 1 three possible lattices are shown. They might correspond to following decimation matrices $\mathbf{M}=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$, $\mathbf{M}=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$, $\mathbf{M}=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ respectively. Be aware that, the usage of 'might' above actually indicates that there exists an infinite number of different decimation matrices for each lattice in Fig. 1. In the following, we regard different decimation matrices having the same lattice as the same **downsampling pattern**. It is well-known that any two matrices of the same pattern are related by right-multiplication of an integer unimodular matrix whose determinant is $\pm 1$ [1].

Considering that the integer unimodular decimation matrix is nothing but a rearrangement of the sample points, matrices of the same pattern may share many properties in common [1]. Exploration of infinite number of decimation matrices can be reduced to exploration of some representatives of distinct patterns. The question is how to enumerate and parameterize all these distinct matrices that can facilitate other related research. This work is motivated by the lack of such enumeration in 2-D. For example, in 1-D, there exists filter bank design techniques that are general enough to span *all* downsampling cases (all integers $\leq 2$). However, most 2-D filter bank designs are done for a specific downsampling patterns. In presence of a concrete enumeration and parametrization of 2-D patterns, determining the span of any 2-D design technique will become possible.

For a given decimation matrix $\mathbf{M}=\begin{pmatrix}m_{00} & m_{01}\\ m_{10} & m_{11}\end{pmatrix}$ and a given lattice $x(\mathbf{n})$ ($\mathbf{n} = [n_1, n_2]^T, n_1, n_2$ are integers), the downsampled lattice is $y(\mathbf{n}) = x(\mathbf{M}\mathbf{n})$. Corresponding downsampling pattern can be obtained by repeating vectors from $\mathbf{M}$ [1]. An example is shown in Figure 2. The vertices form a set, which is a downsampling pattern as defined above. If there exist integer matrix $\mathbf{M}'$ and unimodular integer matrix $\mathbf{V}$ such that, $\mathbf{M} = \mathbf{M}'\mathbf{V}$, $\mathbf{M}'$ is regarded equivalent to $\mathbf{M}$, denoted as $\mathbf{M}' \leftrightarrow \mathbf{M}$. Decimation ratio $m = |\mathbf{M}|$. Therefore for the example in Figure 2, $m = 3$.

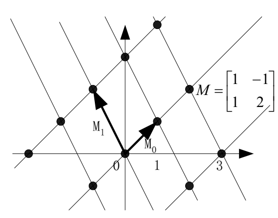


**Fig. 2**. The lattice generated by a decimation matrix $\mathbf{M}$

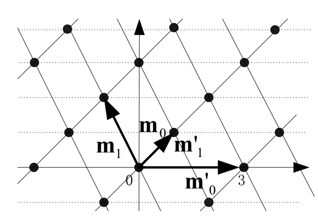

**Fig. 3**. The same lattice for different decimation matrices

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## 2. ENUMERATION AND PARAMETRIZATION OF DOWNSAMPLING PATTERNS

According to the different types of $m$, the discussion has been divided into two sections. One is for decimation ratio $m$ being prime. The other is for $m$ being composite.

### 2.1. Downsampling patterns with prime decimation ratio

For a given decimation matrix $\mathbf{M}=\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$, the following equivalence is always obtainable.

**Lemma 1:** $\mathbf{M} \leftrightarrow \begin{pmatrix} m & n \\ 0 & 1 \end{pmatrix}$ or $\mathbf{M} \leftrightarrow \begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix}$, where $|\mathbf{M}|= m$ is a prime number and $n$ is an integer for some $n$.

Lemma 1 provides normalization to arbitrary decimation matrix. This can be geometrically justified as follows. As we explained above, the pattern is generated by repeating the parallelogram formed by vectors in the decimation matrix. Conversely for this generated lattice, the forming vectors can be chosen as needed. Here since $m$ is prime, they can always be chosen to be $\{\begin{bmatrix} m \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 1 \end{bmatrix}\}$ or $\{\begin{bmatrix} 1 \\ n \end{bmatrix}, \begin{bmatrix} 0 \\ m \end{bmatrix}\}$.

For example, as in Fig. 2, we have the decimation matrix $\mathbf{M}$ $=[\mathbf{M}_0, \mathbf{M}_1]=\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$. From this lattice we can also get another two vectors $\mathbf{m}_0'$ and $\mathbf{m}_1'$. If we choose the vectors as above, we obtain $\mathbf{M}'=[\mathbf{M}_0', \mathbf{M}'_1]=\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. The old and the new decimation matrices are related by $\mathbf{V}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}=\mathbf{M}'$. In Figure 3, the dashed lines denote the new parallelogram generated by $\mathbf{M}'$. Note that half of the dashed lines is overlapped by the original lines generated by $\mathbf{M}$. Although the parallelograms are different, the resampled points are the same, namely the pattern is the same.

**Lemma 2:** $\begin{pmatrix} 1 & 0 \\ m+n & m \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix}$ and $\begin{pmatrix} m & m+n \\ 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} m & n \\ 0 & 1 \end{pmatrix}$

The corresponding unimodular matrices for the above equivalences are trivially $\mathbf{V}=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{V}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

This lemma further reduce our choice of $n$ in lemma 1. It is sufficient to require integer $n$ being within $[0, m-1]$ in order to represent all downsampling patterns. Up to this stage, the number of patterns is reduced to $2m$.

**Lemma 3:** For a given $n \in [1, 2 \cdots m-1]$ there always exists $n'$ such that $\begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix} \leftrightarrow \begin{pmatrix} m & n' \\ 0 & 1 \end{pmatrix}$.

*Proof*: In a Galois field $GF(m)$, every element, except zero, has an inverse. Hence, if $b \in GF(m) (b \neq 0)$, then its inverse is defined as $b^{-1}$ and $bb^{-1} = 1$ [9]. All integers that are less than a prime number $m$ form such a Galois field, and every integer $n < m$ has corresponding inverse $n'$ satisfying $n \cdot n' = 1$ ( mod $m$).

For matrices in Lemma 3, they are related by

$$\begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix} = \begin{pmatrix} m & n' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1-n'n}{m} & -n' \\ n & m \end{pmatrix}.$$

Since $n'$ is the inverse of $n$ as defined above, $\frac{1-n'n}{m}$ is an integer. Hence the unimodular matrix in the right hand side is also an integer matrix. This justifies the equivalence in this lemma. ∎

Lemma 3 rules out the remaining redundancy in the decimation matrices. According to it, matrices in the form of $\begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix}$ can be adopted into matrices in the form of $\begin{pmatrix} m & n \\ 0 & 1 \end{pmatrix}$ or vice versa, except the case with $n = 0$. Therefore up to this step, the number of representative matrices is reduced to $m + 1$. In a fashion of disproving the contradictory (similar to Lemma 5), it may be shown that these $m + 1$ matrices are indeed distinct. Naturally the following theorem is obtained.

**Theorem 1:** For a decimation matrix with a prime determinant of $m$, there only exist $m + 1$ possible different downsampling patterns. The corresponding decimation matrices can be parameterized as $\begin{pmatrix} 1 & 0 \\ n & m \end{pmatrix}$ ($n \in [0, m-1]$) and $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$.

Hence the downsampling patterns in Fig. 1 are the only 3 possible lattices for decimation ratio $m = 2$ and the quincunx matrix is the only non-separable one (whose representative matrix as in Theorem 1 is not diagonal).

*Remark*: As a matter of fact, Theorem 1 can be also interpreted in the following way. First, connect samples with paralleling straight lines. Decimation is performed by retaining only 1 line of samples out of consecutive $m$ lines and discarding the others. For different patterns, these lines incline in different possible angles. Note that this explanation is only valid for prime decimation ratio.

### 2.2. Downsampling patterns with composite decimation ratio

When the decimation ratio is composite, though the above conclusion is no longer valid, a similar approach can be followed. The following lemma is necessary to relate this problem to the problem having been solved.

**Lemma 4:** Given an integer matrix $\mathbf{A}$, it can be factorized into integer matrices as $\mathbf{A} = \prod_{i=1}^{k} \mathbf{A}_i$, if and only if $d = \prod_{i=1}^{k} d_i$, where $d = det(\mathbf{A})$ and $d_i = det(\mathbf{A}_i)$.

*Proof*: ($\Leftarrow$) If there exists a factorization $\mathbf{A} = \prod_{i=1}^{k} \mathbf{A}_i$, equation $d = \prod_{i=1}^{k} d_i$ can be easily obtained by taking the determinant of both sides.

($\Rightarrow$) As we already know, any $D \times D$ integer matrix $\mathbf{M}$ can be factorized as $\mathbf{M} = \mathbf{U\Lambda V}$ [1],[10], where $\mathbf{U}$ and $\mathbf{V}$ are integer unimodular matrices and $\mathbf{\Lambda}$ is a diagonal integer matrix. For our case, $\mathbf{\Lambda}$ is expressible as $\mathbf{\Lambda}=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Since $\lambda_1$ and $\lambda_2$ are also integers, $\mathbf{\Lambda}$ can be further decomposed as

$$\mathbf{\Lambda}=\prod_{i \in S_1} \begin{pmatrix} d_i & 0 \\ 0 & 1 \end{pmatrix} \prod_{j \in S_2} \begin{pmatrix} 1 & 0 \\ 0 & d_j \end{pmatrix}$$

where $\lambda_{1(2)} = \prod_{i \in S_1(S_2)} d_i$ and $S_1 \cup S_2 = \{1, 2, \cdots k\}$. Matrices $\mathbf{A_i}$ are readily obtained. ∎

*Remark*: Note that $\lambda_1$ must be chosen in such a way that it is a factor of $\lambda_2$ [1],[10]. Therefore, if the determinant of decimation matrix $\mathbf{M}$ is $d = a^2 b$, $\lambda_1 = a$ and $\lambda_2 = ab$. Obviously, this factorization sometimes is not unique. For example, when $d = 4$, both $a = 2, b = 1$ and $a = 1, b = 4$ are legal. But $a = 1, b = 6$ is uniquely legal when $d = 6$. Obviously, we can obtain factor matrices $\mathbf{A}_i$ with determinants as prime numbers by choosing all $d_i$ to be prime.

Based on this, the total number of downsampling patterns is given as follows.

**Theorem 2:** For a given composite decimation ratio $m$, if we assume that $m = p_1 p_2 \cdots p_t$, where $p_i$ ($i = 1, \cdots t$) is prime, the total number of downsampling patterns is at most $\prod_{i=1}^{t}(p_i + 1)$. They can be parameterized as $\mathbf{M}=\prod_{i=1}^{t} \mathbf{M}_i(p_i)$, where

$$\mathbf{M}_i(p_i) = \begin{pmatrix} 1 & 0 \\ n_i & p_i \end{pmatrix} \text{ or } \begin{pmatrix} p_i & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad n_i = 0, \cdots, p_i - 1$$

*Proof*: The total number is a consequence of the proposed parametrization. Therefore the parametrization is considered first. For a given decimation matrix $\mathbf{M}$, as we demonstrated in Lemma 4 and its following remark, we can always factor $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{U}\mathbf{M}'_{\mathbf{1}} \cdots \mathbf{M}'_{\mathbf{t}}\mathbf{V}$$

with all matrices $\mathbf{M}'_{\mathbf{i}}$ having prime determinants. Assume that $m = |\mathbf{M}| = p_1 p_2 \cdots p_t$. To make the factorization as unique as possible, we further require $p_1 \leq p_2 \cdots \leq p_t$. Of course, other method of sorting $p_i$ is also applicable once it is consistent in the following. Since $\mathbf{M}'_{\mathbf{i}}$ are diagonal and hence commutative, we can rearrange them as

$$\mathbf{M} = \mathbf{U}\mathbf{M}^{\diamond}_{\mathbf{1}} \cdots \mathbf{M}^{\diamond}_{\mathbf{t}}\mathbf{V}$$

such that $det(\mathbf{M}^{\diamond}_{\mathbf{i}}) = p_i$. We start from the most left item. Since $\mathbf{U}$ is an integer unimodular matrix and $\mathbf{M}^{\diamond}_{\mathbf{1}}$ is an integer matrix with prime determinant, according to Theorem 1, $\mathbf{U}\mathbf{M}^{\diamond}_{\mathbf{1}}$ can be always expressed as $\mathbf{M}_1 U_1$ where $\mathbf{M}_1$ is identically in the form of $\begin{pmatrix} 1 & 0 \\ n_1 & p_1 \end{pmatrix}$ or $\begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix}$ with $n_1 = 0, \cdots, p_1 - 1$ and $\mathbf{U}_1$ is also an integer unimodular matrix. Iteratively $\mathbf{U_1}\mathbf{M}^{\diamond}_{\mathbf{2}} \leftrightarrow \mathbf{M_2}\mathbf{U_2}$ and so forth. Finally the decimation matrix can be written as

$$\mathbf{M} = \mathbf{M_1} \cdots \mathbf{M_t}\mathbf{U}_t\mathbf{V}.$$

It is indeed one of the patterns in the proposed parametrization.

The parametrization may repeat some patterns. That is why the term 'at most' is used in the theorem. Under such circumstance, the theorem provides an upper bound of the number of downsampling patterns. ∎

## 3. EXAMPLES OF DOWNSAMPLING PATTERNS

Apart from the $m = 2$ example in the beginning, more examples to illustrate Theorem 1 and Theorem 2 are given in this section.

For decimation ratio $m = 3$, according to Theorem 1, there exists 4 different downsampling patterns. Two nonseparable patterns among them are shown in Fig. 4. The remark of Theorem 1 involving straight line interpretation is also reflected in this figure.

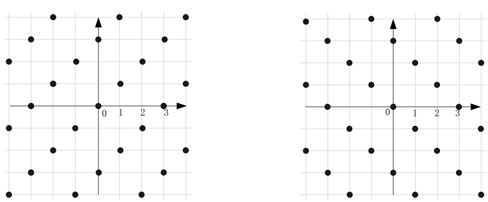


**Fig. 4**. Two nonseparable patterns for $|\mathbf{M}| = 3$

| | $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ |
|---|---|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ ⓐ | $\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ ⓑ | $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ⓒ |
| $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & 4 \end{pmatrix}$ ⓓ | $\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$ ⓔ | $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$ ⓒ |
| $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ⓒ | $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ ⓕ | $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ ⓖ |

**Table 1**. Parametrization of downsampling patterns when $m = 4$

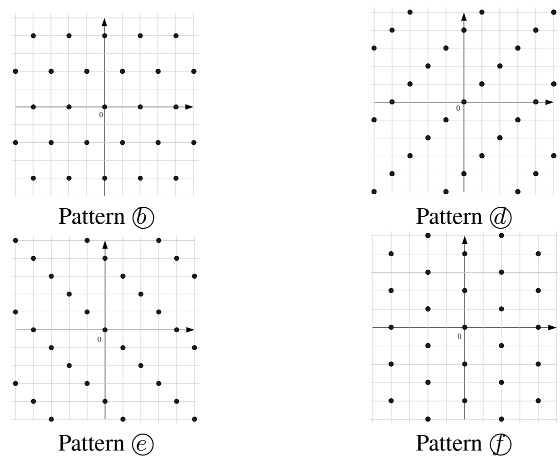


**Fig. 5**. Four nonseparable patterns for $|\mathbf{M}| = 4$

For decimation ratio $m = 4$, according to Theorem 2, since $m = 2 \times 2$, there should exist at most 9 different downsampling patterns. Actually, given that both factors of $m$ are 2, there are some identical patterns. After ruling out the repetition, there exists 7 different patterns and the corresponding decimation matrices can be parameterized by the product of 2 matrices with determinant of 2, as shown in Table 1. Possible choices for $\mathbf{M_1}$ and $\mathbf{M_2}$ as in Theorem 2 are listed in the first column and in the first row respectively. Different circled letters are also used for indicating distinct patterns. Note that 3 different matrices correspond to the same patterns ⓒ . Four nonseparable patterns are also depicted in Fig. 5.

Similarly, the parametrization for decimation ratio $m = 6$ is given in Table 2. Since the factors are different, there are exactly 12 different patterns. Some readers may be curious about the consequence of interchanging $\mathbf{M_1}$ and $\mathbf{M_2}$. The resulting patterns are exactly the same as those in Table 2, yet in a different arrangement. Recalling the proof of Theorem 2, this equivalence is, in fact, guaranteed by the commutativity of diagonal matrices.

| | $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ | $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ |
|---|---|---|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ ⓐ | $\begin{pmatrix} 1 & 0 \\ 2 & 6 \end{pmatrix}$ ⓑ | $\begin{pmatrix} 1 & 0 \\ 4 & 6 \end{pmatrix}$ ⓒ | $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ ⓓ |
| $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$ ⓔ | $\begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix}$ ⓕ | $\begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix}$ ⓖ | $\begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}$ ⓗ |
| $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ ⓘ | $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ ⓙ | $\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$ ⓚ | $\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$ ⓛ |

**Table 2**. Parametrization of downsampling patterns when $m = 6$

For greater decimation ratio, the previous results should be utilized to avoid the repeating of patterns. For example, when $m = 12$, the solutions for $m = 3$ and $m = 4$ should be used. We list the numbers of different patterns for some decimation ratio in Table 3. In the result, the repeating cases have been ruled out by exhaustive checking.

## 4. ENUMERATION OF NONSEPARABLE PATTERNS

In many applications, people are willing to trade a better performance with an increased complexity especially when the cost for implementation is more and more inexpensive. Nonseparable filter bank is such a case in point. Though it requires more resource in terms of both memory and computation, the better performance over its separable counterpart still makes it attractive to many researchers. For this kind of filter bank, the nonseparable downsampling patterns are essential. We further propose a specific approach to parameterizing all nonseparable patterns.

| m | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ♯ of Pat. | 3 | 4 | 7 | 6 | 12 | 8 | 15 | 13 | 18 | 12 | 28 |

**Table 3**. Numbers of different patterns for some decimation ratio

**Lemma 5:** Patterns with decimation matrices $\mathbf{M}=\begin{pmatrix}1 & 0\\ n & m\end{pmatrix}$ or $\mathbf{M}=\begin{pmatrix}m & n\\ 0 & 1\end{pmatrix}$ for any integer $m$ and $n = 1, \cdots, m-1$ are nonseparable.

*Proof*: According to this lemma, we can not find an integer unimodular matrix $\mathbf{U}$ such that $\begin{pmatrix}m & n\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\cdot\mathbf{U}$ with integers $a$ and $b$ satisfying $m=ab$. Obviously, if such matrix exists, it can be expressed as

$$\mathbf{U}=\begin{pmatrix}\frac{1}{a} & 0\\ 0 & \frac{1}{b}\end{pmatrix}\begin{pmatrix}m & n\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{m}{a} & \frac{n}{a}\\ 0 & \frac{q}{b}\end{pmatrix}.$$

To make $\mathbf{U}$ integer, the only choice is $a = m, b = 1$. This choice disables the entry $\frac{n}{a}$ from integer since $n \in [1, m-1]$. Similar proof goes for $\begin{pmatrix}1 & 0\\ n & m\end{pmatrix}$. ∎

*Remark*: From Theorem 1 and Lemma 5, the number of distinct nonseparable patterns for prime $m$ is $m-1$ and these patterns can be parameterized as $\begin{pmatrix}1 & 0\\ n & m\end{pmatrix}, n \in [1, m-1]$.

**Theorem 3:** For a decimation ratio $m = p_1p_2$, where prime numbers $p_1 \neq p_2 \neq 1$, its corresponding downsampling patterns that are nonseparable can be parameterized as

$$\mathbf{M}=\begin{pmatrix}1 & 0\\ n & p_1p_2\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}p_1p_2 & n\\ 0 & 1\end{pmatrix} \quad \text{with } n = 1, \cdots p_1p_2 - 1 \; .$$

*Proof*: From Lemma 5, both $\mathbf{M}=\begin{pmatrix}1 & 0\\ n & p_1p_2\end{pmatrix}$ and $\begin{pmatrix}p_1p_2 & n\\ 0 & 1\end{pmatrix}$ are nonseparable. As shown in Table 1 and Table 2, the patterns within the first $p_1$ rows and the first $p_2$ columns are of the form

$$\begin{pmatrix}1 & 0\\ n_1 & p_1\end{pmatrix}\begin{pmatrix}1 & 0\\ n_2 & p_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ n_1+n_2p_1 & p_1p_2\end{pmatrix}.$$

Since $p_1$ and $p_2$ are coprime, for $n_1 \in [0, p_1-1]$ and $n_2 \in [0, p_2-1]$, $n_1 + n_2p_1 \in [0, p_1p_2-1]$. Thus, all these $p_1p_2$ patterns may be parameterized in the form of $\begin{pmatrix}1 & 0\\ n & p_1p_2\end{pmatrix}$. For $n \in [1, p_1p_2-1]$, they are nonseparable.

This leaves $p_2 - 1$ matrices in the last row of the table (except first and last entries, which are separable) and $p_1 - 1$ matrices in the last column (except first and last entries). Decimation matrices in the last column can be expressed as $\begin{pmatrix}1 & 0\\ n' & p_1\end{pmatrix}\begin{pmatrix}p_2 & 0\\ 0 & 1\end{pmatrix}$ where $n' \in [1, p_1-1]$. We now show each such matrix is equivalent to a matrix of the form given in the theorem, or that there exists an integer unimodular matrix $\mathbf{U}$ such that

$$\begin{pmatrix}1 & 0\\ n' & p_1\end{pmatrix}\begin{pmatrix}p_2 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}p_1p_2 & n^\diamond\\ 0 & 1\end{pmatrix}\mathbf{U}.$$

It follows that

$$\mathbf{U}=\begin{pmatrix}\frac{1}{p_1p_2} & \frac{n^\diamond}{p_1p_2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ n' & p_1\end{pmatrix}\begin{pmatrix}p_2 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1-n'n^\diamond}{p_1} & \frac{-n^\diamond}{p_2}\\ p_2n' & p_1\end{pmatrix}.$$

The task is reduced to showing that there exists such $n^\diamond$. Following Bezout Identity, we can always find integers $k, j$ for the given coprimes $p_1, p_2$ such that $p_2k - p_1j = 1$.

Following the proof of Lemma 3, we can always find an integer $n^*$ within $[1, p_1-1]$ such that $\frac{1-n'n^*}{p_1}$ is an integer. Now choose $n^\diamond = n^* + p_1(n^*j)$. It follows that $\frac{1-n'n^\diamond}{p_1}$ is indeed an integer. Further, since $n^*p_2k - n^*p_1j = n^*$, this choice of $n^\diamond$ makes sure that $\frac{n^\diamond}{p_2} = n^*k$ is also an integer.

Similar approach may be used for the matrices in the last row. ∎

*Remark*: For composite decimation ratio, the above parametrization is complete yet redundant, which means repeating may happen. For example, for any integer $m$, (not necessarily prime)

$$\begin{pmatrix}m & m-1\\ 0 & 1\end{pmatrix}\leftrightarrow\begin{pmatrix}1 & 0\\ m-1 & m\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}m & 1\\ 0 & 1\end{pmatrix}\leftrightarrow\begin{pmatrix}1 & 0\\ 1 & m\end{pmatrix} \; .$$

As emphasized in the theorem, $m$ is necessary to be the product of only two distinct prime numbers. Whether this can be extended to more generic case is of further interest.

## 5. CONCLUSION

In this paper, downsampling patterns for arbitrary decimation ratios are studied. When the decimation ratio $m$ is a prime number, there exists $m + 1$ different patterns and $m - 1$ of them are nonseparable. Both these cases can be completely parameterized. When the decimation ratio $m$ is composite and can be expressed as the product of prime numbers $m = p_1p_2 \cdots p_t$, where prime numbers $p_i \neq 1$ ($i = 1, \cdots t$), there exists at most $\prod_{i=1}^{t}(p_i + 1)$ patterns. These patterns can be completely parameterized. Further for $t = 2$ with $p_1 \neq p_2$, a parametrization for all nonseparable patterns has been proposed.

## 6. REFERENCES


[1] P.P. Vaidyanathan, *Multirate systems and filter banks*, Englewood Cliffs: Prentice Hall, 1993.

[2] T. Chen, P.P Vaidyanathan, "Commutativity of D-dimensional decimation and expansion matrices, and application to rational decimation systems," *Proceeding, ICASSP*, vol. 4, pp. 637–640, March 1992.

[3] T. Chen, P.P Vaidyanathan, "Least common right/left multiples of integer matrices and applications to multidimensional multirate systems," *Proceeding, ISCAS*, vol. 2, pp. 935–938, May 1992.

[4] A.A.C.M. Kalker, "Commutativity Of Up/Down Sampling," *Electronics Letters*, vol. 28, no. 6, pp. 567–569, March 1992.

[5] J. Kovacevic, M. Vetterli, "The commutativity of up/downsampling in two dimensions," *IEEE TRANS. on Information Theory*, vol. 37, no. 3(II), pp. 695–698, May 1991.

[6] R.A. Gopinath, C.S. Burrus, "On Upsampling, Downsampling and Rationanl Sampling Rate Filter Banks," *IEEE TRANS. on Signal Processing*, vol. 42, no. 4, pp. 812–824, April 1994.

[7] B.L. Evans, R.H. Bamberger, J.H. Mcclellan, "Rules for Multidimensional Multirate structures," *IEEE TRANS. on Signal Processing*, vol. 42, no. 4, pp. 762–770, April 1994.

[8] T. Chen, P.P Vaidyanathan, "Recent Developments in Multidimensional Multirate Systems," *IEEE TRANS. on Circuits and Systems for Video Technology*, vol. 3, no. 2, pp. 116–137, April 1993.

[9] J.G. Proakis, *Digital communications, 4ed*, McGraw-Hill, 2001.

[10] H.J.S. Smith, "On System Of Linear Indeterminate Equations And Conguences," *Philos. Trans. Royal Soc. London*, vol. 151, pp. 293–326, 1861.