SAMPLED SIGNAL RECONSTRUCTION VIA H^2 OPTIMIZATION

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ABSTRACT

In this paper the sampled signal reconstruction problem is formulated and solved as the sampled-data H^2 smoothing problem, in which an analog reconstruction error is minimized. Both infinite (non-causal reconstructors) and finite (reconstructors with relaxed causality) preview cases are considered. The optimal reconstructors are in the form of the cascade of a discrete-time smoother and a generalized hold (interpolator). In the particular case of reconstructing polynomial signals with infinite preview, the proposed procedure recovers the cardinal B-spline reconstructors.

1. INTRODUCTION AND PROBLEM FORMULATION

This paper studies the problem of reconstructing an analog signal from sampled noisy measurements. The problem is presented by the scheme in Fig. 1, where v is the analog signal to be reconstructed and $\bar{y}$ is the discrete measured signal, which is the sampling (with the ideal sampling device S_h with the sampling period h) of an analog signal y corrupted by a discrete noise $\bar{n}$ (which can also be used to account for quantization errors). Our goal here is to design a hybrid, digital/analog, reconstructor (estimator) $\mathcal{K}$ generating an estimate $\hat{v}$ of v so that the estimation error $e = v - \hat{v}$ is "small" (in a sense to be defined latter). As is conventional, we allow the estimator to have access to "future" measurements, either the whole future (fixed-interval smoothing formalism) or within a fixed-length window (fixed-lag smoothing formalism). The latter is equivalent to reconstructing a delayed version of v.

Note that the signals v and y do not need to be the same. This does not complicate the solution while it makes the setup more flexible. For example, we can formulate the problem of reconstructing continuous-time velocity from sampled position measurements (or, equivalently, acceleration from sampled velocity measurements), in which case $v = \dot{y}$. We assume that both v and y are modeled as outputs of a given continuous-time system $\mathcal{G}$ driven by a common input w, which can include fictitious inputs used to model v and y, continuous-time disturbances, etc. The signal generator $\mathcal{G}$, which is given in terms of its transfer matrix

$$G(s) = \begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = \begin{bmatrix} C_v \\ C_y \end{bmatrix} (sI - A)^{-1} B, \tag{1}$$

is used here to account for properties of v and y, such as their spectral densities, cross correlation, bandwidth, etc. We also assume that $\bar{n} = \Phi^{1/2} \bar{w}_n$, where $\bar{w}_n$ is a unit-variance white noise sequence and the matrix $\Phi \geq 0$ is the spectral density of $\bar{n}$.

The reconstruction problem is typically treated in a non-model-based fashion, i.e., without modeling the signals v and y. For example, one may fix a continuous-time interpolant (the D/A circuit), like polynomial splines, and then add a digital filter to guarantee the consistency of the reconstruction, see [1] and the references therein. The model-based approach, under the assumption that the D/A circuit is the zero-order hold followed by a known analog postfilter, was introduced in [2], where H^∞ optimization was proposed to determine the digital part of $\mathcal{K}$. The common shortcoming of the available solutions is their limited ability to handle (non)causality constraints on $\mathcal{K}$. The spline-based techniques produce fixed-interval reconstructors, the impulse response of which is then truncated to produce fixed-lag solutions. This approach can be justified only when the fixed-interval solutions decay rapidly, so that the decay rate becomes an important factor in the choice of continuous-time interpolants. This might compromise the reconstruction performance. The model-based design of [2, 3] addresses the fixed-lag situation by considering the reconstruction of the delayed signal. Yet the delay there is embedded into $\mathcal{G}$, which leads to solutions whose computational burden grows rapidly with the increase of the preview length and also obscures the problem structure.

In this paper we show that the reconstruction problem can be efficiently addressed in the H^2 (least-mean-squares) framework. Our purpose here is twofold: we demonstrate that closed-form solutions, the computation burden of which do not depend on the length of preview, can be derived and that the model-based problem formulation can result in easily interpretable solutions, containing as particular cases some known results, such as B-spline reconstructors [4]. More specifically, in deriving our results we make use of the lifting transformation [5] and the technique of [6] (although because of the space limitations we omit most derivation details). The optimal reconstructors enjoy the continuity and, when $v = y$ and $\Phi = 0$ (i.e., no measurement noise), consistency properties. Moreover, in the special case when $G_v(s) = G_y(s) = 1/s^m$ and $\Phi = 0$ we recover the standard B-spline solutions of [4] in the fixed-interval setup.

As mentioned above, we adopt the H^2 formalism in measuring the reconstruction performance. This means that we measure it by the H^2 norm, denoted as $\|\cdot\|_2$, of the error system

$$\mathcal{G}_{\mathrm{e}} \doteq \begin{bmatrix} \mathcal{G}_v & 0 \end{bmatrix} - \mathcal{K} \begin{bmatrix} S_h \mathcal{G}_y & \Phi^{1/2} \end{bmatrix}$$

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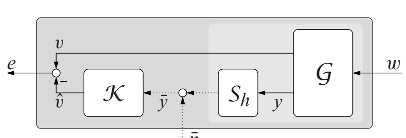


Fig. 1. The problem setup

from the exogenous signals w and $\bar{w}_n$ to the reconstruction error e (the dark box in Fig. 1). Although this is a hybrid and periodically time-varying system, its H^2 norm is well defined through the lifted-domain representation [7, 8]. The stochastic interpretation of this norm is the averaged variance of e when w and $\bar{w}_n$ are uncorrelated zero-mean continuous- and discrete-time white Gaussian processes with unitary covariances (least mean-square estimation). The deterministic interpretation is the energy (L^2 norm) of e, averaged over one sampling period $\tau \in [0, h]$, when $w(t) = \delta(t - \tau)$ and $\bar{w}_n$ is the unit pulse.

Thus, the problem we address is the minimization of $\|\mathcal{G}_{\rm e}\|_2$ by a stable (i.e., bounded as an operator from ℓ^2 to L^2) estimator $\mathcal{K}$. We also require that the operator $\mathcal{D}_{hl}\mathcal{K}$ be causal, where the notation $\mathcal{D}_\tau$ stands for the τ-delay operator with the transfer function $\mathrm{e}^{-s\tau}$. This is equivalent to allowing $\mathcal{K}$ to use previewed measurements, with the preview length of l steps. When l, called the *smoothing lag*, is finite, the estimation problem is referred to as the *fixed-lag smoothing*. When $l = \infty$, in which case no causality constraints are imposed, the problem is referred to as the *fixed-interval smoothing*. Throughout the paper we assume that the state-space realization in (1) is minimal, C_y has full row rank, and the pair (C_y, e^{Ah}) is detectable, i.e., has no hidden modes outside the open unit disk.

2. FIXED-INTERVAL SOLUTION

We start with the fixed-interval case in which no causality constraints are imposed on $\mathcal{K}$. All proofs and derivation details are omitted because of space the limitations.

2.1. Solution

Define the matrix exponential function

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ 0 & \Sigma_{22}(t) \end{bmatrix} = \exp\left(\begin{bmatrix} A & -BB' \\ 0 & -A' \end{bmatrix} t\right),$$

where $\Sigma_{11}(t) = \mathrm{e}^{At}$ and $\Sigma_{12}(t) = -\int_0^t \mathrm{e}^{A\tau} BB' \mathrm{e}^{A'\tau} \mathrm{d}\tau\, \mathrm{e}^{-A't}$. To simplify the notation, we omit the parentheses when $t = h$, so that we write Σ instead of $\Sigma(h)$.

Let Y be the stabilizing solution to the following algebraic Riccati equation (ARE):

$$Y = \Sigma_{11}\big(Y - \Sigma_{11}^{-1}\Sigma_{12} - YC_y'(\Phi + C_y Y C_y')^{-1} C_y Y\big)\Sigma_{11}'. \quad (2)$$

This is the standard ARE associated with the sampled-data filtering for $\mathcal{G}$ [5] and under our assumptions (minimality of the realization in (1), C_y has full row rank, and (C_y, e^{Ah}) is detectable) the stabilizing solution such that $Y > 0$ and $\det(\Phi + C_y Y C_y') \neq 0$ always exists (even for $\Phi = 0$). By stabilizing we mean that the matrix

$$\bar{A}_L \doteq \big(I - YC_y'(\Phi + C_y Y C_y')^{-1} C_y\big)\Sigma_{11}$$

is Schur (has all eigenvalues inside the open unit disk). Now, let $P \geq 0$ be the solution to the Lyapunov equation

$$P = \bar{A}_L' P \bar{A}_L + \Sigma_{11}' C_y' (\Phi + C_y Y C_y')^{-1} C_y \Sigma_{11}. \quad (3)$$

The main result of this section is formulated as follows:

Theorem 1. *The unique solution of the fixed-interval reconstruction problem is brought by the estimator $\mathcal{K} = \mathcal{K}_c + \mathcal{K}_{ac}$, where the causal part $\mathcal{K}_c$ is the cascade of the digital filter with the transfer function*

$$\bar{K}_c(z) = z(zI - \bar{A}_L)^{-1} Y C_y' (\Phi + C_y Y C_y')^{-1}$$

and the hold function

$$\phi_c(t) = \begin{bmatrix} C_v & 0 \end{bmatrix} \Sigma(t) \begin{bmatrix} I - Y_a P \\ P \end{bmatrix}, \quad t \in [0, h),$$

and the anti-causal part $\mathcal{K}_{ac}$ is the cascade of the digital filter with the transfer function

$$\bar{K}_{ac}(z) = (z^{-1}I - \bar{A}_L')^{-1}(\Sigma_{11}' - \bar{A}_L' P Y) C_y' (\Phi + C_y Y C_y')^{-1}$$

and the hold function

$$\phi_{ac}(t) = \begin{bmatrix} C_v & 0 \end{bmatrix} \Sigma(t) \begin{bmatrix} Y_a \\ -I \end{bmatrix}, \quad t \in [0, h),$$

where $Y_a \doteq Y - YC_y'(\Phi + C_y Y C_y')^{-1} C_y Y \geq 0$.

The generalized hold acts as follows. Let $\bar{K}_{\rm c}$ generate the sequence $\{\bar{x}_k\}$, which has the same dimension as the state vector of $\mathcal{G}$. Then the continuous-time output of $\bar{K}_{\rm c}$, $\hat{v}_{\rm c}(t)$, is calculated as

$$\hat{v}_{\rm c}(kh + \tau) = \phi_{\rm c}(\tau)\bar{x}_k, \quad \forall \tau \in [0, h), k \in \mathbb{Z}^+.$$

The anti-causal part behaves similarly.

To calculate the optimal achievable performance, i.e., the minimal $\|\mathcal{G}_{\rm e}\|_2$, we need to introduce the matrix

$$\begin{bmatrix} \Sigma & \Delta \\ 0 & \Sigma \end{bmatrix} \doteq \exp\left(\left[\begin{array}{cc:cc} A & -BB' & 0 & 0 \\ 0 & -A' & -C_v'C_v & 0 \\ \hdashline 0 & 0 & A & -BB' \\ 0 & 0 & 0 & -A' \end{array}\right] h\right).$$

Then the following result can be formulated

Lemma 1. *The optimal performance attained by the estimator in Theorem 1 is*

$$J_{FI} = \sqrt{\frac{1}{h}\,\mathrm{tr}\left(\begin{bmatrix} -P & I - PY_a \end{bmatrix} \Sigma^{-1} \Delta \begin{bmatrix} -Y_a \\ I \end{bmatrix}\right)},$$

where Y_a is as defined in Theorem 1.

2.2. Properties of the solution

The impulse response of the optimal estimator, $k(t)$, is its response to the discrete unit pulse applied at the zero time instance, $\bar{\delta}_i$. This response is a continuous-time signal, which completely determines the estimator. Below, some properties of $k(t)$ are presented.

Proposition 1 (Continuity). *The function $k(t)$ is continuous.*

Proposition 2 (Consistency). *Let $C_v = C_y$ and $\Phi = 0$. Then the sampled impulse response $k(ih) = \bar{\delta}_i$.*

Continuity of $k(t)$ implies that the resonstructed signal, $\hat{v}$, is also continuous. Consistency means that $\hat{v}$ at the sampling instances ih equals the samples $\bar{y}$ (which are the samples of v in this case) on which $\hat{v}$ is based.

For finite-dimensional signal generators $\mathcal{G}$, the function $k(t)$ is a linear combination of piecewise exponential functions, possibly including piecewise polynomials. It turns out that it is purely piecewise polynomial if $\mathcal{G}$ consists only of integrators (in effect, the proposition below states that we recover the cardinal B-splines of [4]):

Proposition 3 (Polynomial splines)**.** *Let $G_v(s) = G_y(s) = 1/s^m$ and $\Phi = 0$. Then the optimal estimator $k(t)$ is the unique stable $(2m-2)$-smooth $(2m-1)$-degree polynomial spline for which $k(ih) = \bar{\delta}_i$.*

Another important property of the optimal reconstruction is:

Proposition 4 (Analog performance recovery)**.** *Let $\Phi = 0$. Then J_{FI} recovers the performance of the continuous-time reconstruction as $h \to 0$. In particular, when $G_v(s) = G_y(s)$, $\lim_{h\to 0} J_{FI} = 0$.*

A possible alternative to the H^2 formalism adopted in this paper is the H^∞ approach [2, 5], in which an induced (minimax) L^2/ℓ^2 norm of the error system $\mathcal{G}_e$ is minimized. A remarkable property of the fixed-interval H^2 solution is that it is actually H^∞ optimal as well.

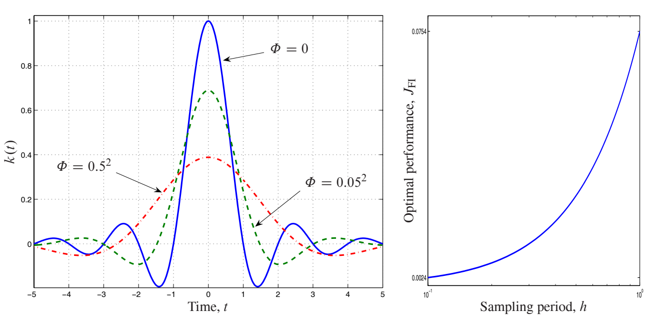
Proposition 5 (H^∞ optimality)**.** *Estimator $\mathcal{K}$ in Theorem 1 minimizes $\|\mathcal{G}_e\|_\infty$, i.e., it attains the minimal γ for which*

$$\|e\|^2_{L^2} \le \gamma^2\big(\|w\|^2_{L^2} + \|\bar{w}_n\|^2_{\ell^2}\big)$$

for all $w \in L^2(\mathbb{R})$ and $\bar{w}_n \in \ell^2(\mathbb{Z})$.

2.3. Illustrative examples

To illustrate the algorithm of §2.1 and properties of the resulting reconstructors, consider several simple examples.



(a) Impulse response of $\mathcal{K}$, $h = 1$ (b) Optimal $\|\mathcal{G}_e\|_2$

Fig. 2. Illustrations for Example 1

Example 1. We start with the reconstruction problem for

$$\begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = \begin{bmatrix} 1/s^2 \\ 1/s^2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right]\left(sI - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This corresponds to the problem of reconstructing the position of a process, the acceleration of which is modeled as a white noise.

Fig. 2(a) shows the impulse responses of three optimal estimators under $\Phi = 0$ (perfect measurements), $\Phi = 0.05^2$ (small roundoff errors), and $\Phi = 0.5^2$ (larger roundoff errors), all with $h = 1$. Under the perfect measurement conditions the reconstructor is actually the cubic B-spline interpolator combined with a digital filter to guarantee the consistency of the reconstruction [4]. The estimator is indeed consistent, see the solid line in Fig. 2(a), which is zero whenever $t = ih$, $i \ne 0$. When the measurements are noisy, the reconstructors become smoother, yet are no longer consistent (in fact, $S_h\mathcal{K}$ is the Kalman smoother for the discretized process). The optimal performance versus the sampling period h is presented in Fig. 2(b) for the case of $\Phi = 0$. As expected, the optimal $\|\mathcal{G}_e\|_2$ is an increasing function of h, which approaches zero as $h \to 0$.

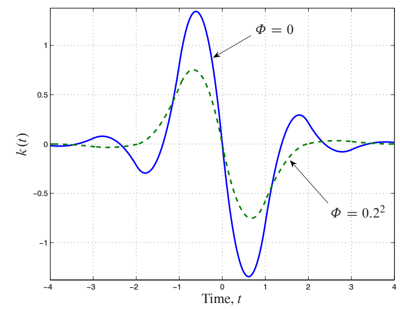
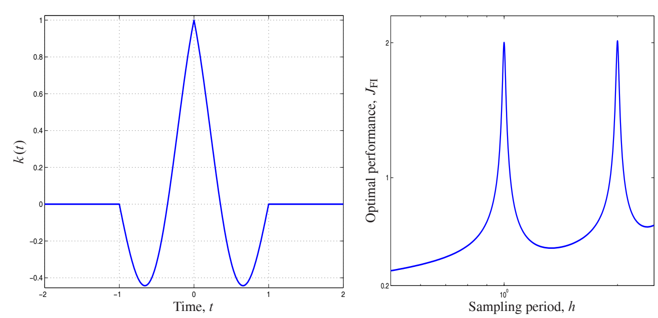


Fig. 3. Illustrations for Example 2

Example 2. As mentioned in the Introduction, our setup is well suited to reconstruct more complicated functions of y. To illustrate this point, consider the problem, similar to that in Example 1, yet where the signal to be reconstructed is the velocity of y. This requires $G_v(s) = sG_y(s)$, so that the matrix C_v in the state-space realization of Example 1 should be replaced with $\begin{bmatrix} 0 & 1 \end{bmatrix}$. The optimal estimators for $\Phi = 0$ and $\Phi = 0.2^2$ are shown in Fig. 3. These curves can be thought of as approximate derivatives of the δ-function. Similarly to what we saw in Example 1, the increase of Φ makes the reconstructors smoother.



(a) Impulse response of $\mathcal{K}$, $h = 1$ (b) Optimal $\|\mathcal{G}_e\|_2$

Fig. 4. Illustrations for Example 3

Example 3. Now consider the reconstruction problem for

$$G_v(s) = G_y(s) = \frac{s}{(s + 0.01\pi)^2 + \pi^2},$$

which represents the reconstruction of a signal with a dominant harmonic at $\omega = \pi$. Fig. 4(a) shows the optimal reconstructor for $h = 1$ and $\Phi = 0$. This $\mathcal{K}$ is virtually an FIR filter (exponential spline). Note that this sampling rate corresponds to the Nyquist frequency of this harmonic, so that the reconstruction problem appears fairly senseless because of a heavy aliasing around the dominant frequency for this h. The purpose of this example is to demonstrate that this situation does show up through a dramatic deterioration of the achievable H^2 performance. Indeed, it is seen at Fig. 4(b), which depicts the optimal performance as a function of the sampling period, that there is a sharp performance peak at $h = 1$. Another peak corresponds to the doubled Nyquist frequency, where the aliasing also mixes the dominant harmonics (actually, there are peaks at each $h \in \mathbb{N}$).

3. FIXED-LAG SOLUTION

In the fixed-lag formulation we assume that $\mathcal{D}_{hl}\mathcal{K}$ is causal, i.e., that the reconstructor has access to measurements l steps ahead. The resulting solutions are, thus, suitable for real-time implementation.

3.1. Solution

The notation used in this section are introduced in §2.1. The main result is formulated as follows:

Theorem 2. *The unique solution of the fixed-lag problem is brought by $\mathcal{K} = \mathcal{K}_c + \mathcal{K}_{ac,tr} + \mathcal{K}_{corr}$, where $\mathcal{K}_c$ is the causal part of the fixed-interval solution, $\mathcal{K}_{ac,tr}$ is the anti-causal part of the fixed-interval solution whose impulse response is truncated to $[-lh, 0]$, and $\mathcal{K}_{corr}$ is the "correction" term, which is the cascade of the digital filter with the transfer function $z^l \bar{K}_c(z)$ and the hold function*

$$\phi_{corr}(t) = \begin{bmatrix} C_v & 0 \end{bmatrix} \Sigma(t) \begin{bmatrix} Y_a \\ -I \end{bmatrix} (\bar{A}_L^l)' P, \quad t \in [0, h).$$

It is worth emphasizing that the causality constraint arises in our treatment as part of the optimization problem. This is a clear advantage over the approach in which causality constraints are imposed *after* the fixed-interval solution is obtained, by truncating the impulse response of the latter. The performance of the resulting solution depends then on the decay properties of the fixed-interval solution. Theorem 2 says that this truncation should be corrected by adding the term $\mathcal{K}_{\rm corr}$. At the same time, the gain of the latter is proportional to $\bar{A}_L^l$, which decays exponentially as l increases (since $\bar{A}_L$ is Schur). Hence, for a sufficiently large (with respect to the dynamics of $\bar{A}_L$) smoothing lag the truncation is justifiable.

Also, we can quantify the deterioration of the achievable performance with respect to the fixed-interval case. We have:

Lemma 2. *The optimal performance attained by the estimator in Theorem 2 is*

$$J_{FL} = \sqrt{J_{FI}^2 + \frac{1}{h}\,\mathrm{tr}\left((\bar{A}_L^l)' P \bar{A}_L^l \begin{bmatrix} I & Y_a \end{bmatrix} \Sigma^{-1} \Delta \begin{bmatrix} -Y_a \\ I \end{bmatrix} \right)},$$

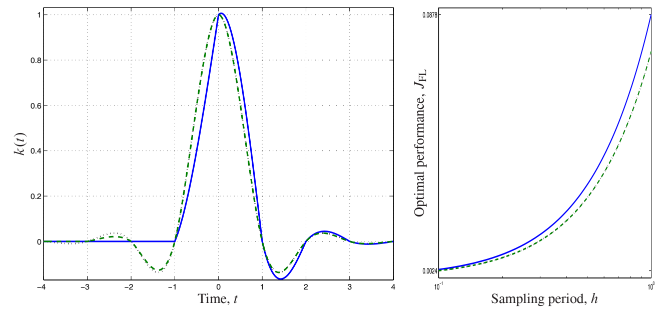
Note that the performance deterioration is proportional to $\bar{A}_L^l$, so that $J_{\rm FL}$ approaches $J_{\rm FI}$ exponentially as $l \to \infty$.

3.2. Properties of the solution

Remarkably, the fixed-lag solution inherits most properties of its fixed-interval version. Namely, for every $l > 0$ the impulse response of the optimal smoother is a continuous function of time (the continuity property does not extend to the filtering case, $l = 0$, though), when $C_v = C_y$ and $\Phi = 0$ the sampled smoother is still the unit pulse (consistency), and when $\Phi = 0$ the continuous-time performance is recovered as $h \to 0$. The fixed-lag H^2 optimal solution, however, is no longer H^∞ optimal.

3.3. Illustrative example

Example 4. Consider again the problem in Example 1 with $\Phi = 0$, now in the fixed-lag setting. Fig. 5(a) presents the impulse responses of the reconstructors designed for $l = 1$ (solid line) and $l = 3$ (dashed line). The former curve is quite different from that for the fixed-interval $k(t)$ (shown by the black dotted line), it is even not differentiable. The latter curve is much closer to the fixed-interval impulse response, especially in its causal part. The comparison via the achievable performance in Fig. 5(b), which is *the* means to compare these reconstructors, reveals even a better match between the fixed-interval and fixed-lag with $l = 3$ solutions—their plots are virtually indistinguishable. The solid curve there, corresponding to $l = 1$, shows some deterioration of the achievable performance.



(a) Impulse response of $\mathcal{K}$, $h = 1$ (b) Optimal $\|\mathcal{G}_e\|_2$

Fig. 5. Illustrations for Example 4

4. CONCLUDING REMARKS

In this paper we have presented a model-based H^2 framework for the problem of reconstructing analog signals from sampled noisy measurements, in which the task is formulated as a sampled-data smoothing problem. The solutions derived in the paper are in the form of exponentional / polynomial splines, which have clear structural properties (such as continuity, consistency, etc) and recover in some special cases known structures. The proposed solutions can also rigorously incorporate constraints on the length of preview available to the reconstructor.

5. REFERENCES

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