UPPER AND LOWER BOUNDS FOR THE THRESHOLD OF THE FFT FILTER BANK-BASED SUMMATION CFAR DETECTOR

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ABSTRACT

The reliable computation of detection threshold T given a desired probability of false alarm P_{fa} is an important issue in the design of the FFT filter bank-based summation CFAR (constant false alarm rate) detector. The computation of detection threshold T is based on numerical procedures such as the Newton-Ralphson algorithm and a priori knowledge of lower and upper bounds for T for a given P_{fa}. Current approaches used in the initialization stage of the computation of threshold T are largely *ad hoc* as there are no theoretical upper and lower bounds for T reported in the literature. In this article, several theoretical upper and lower bounds for T for overlapped and non-overlapped signal data are derived. These results enable a proper design of the FFT filter bank-based summation CFAR detector.

1. INTRODUCTION

The FFT filter bank-based summation CFAR detector is an efficient technique for detecting narrowband signals in noise and has important applications in civilian spectrum monitoring, electronic warfare radio surveillance systems, radio astronomy and instrumentation. This detector operates by forming spectral power estimates in channels, each of which corresponds to a group of one or more contiguous FFT bins, and comparing these power estimates against a detection threshold, T. A signal is declared to exist in that channel only if the power in a channel exceeds T. The performance analysis of the FFT filter bank-based summation CFAR detector has been the subject of much study [1]-[7]. In particular, closed-form algebraic formulas giving the probability of false alarm, P_{fa}, as a function of T have been derived [1], [2], [6]. These results enable T to be computed for a given P_{fa} through numerical procedures. However, practical implementations have been largely *ad hoc* since a good lower or upper bound for T is required for the initialization of the numerical procedures and a usable theoretical solution for the bounds is not available. Note that using good bounds for T in the initialization of the numerical procedures is highly desirable since it reduces the likelihood of problems with numerical errors. Also, the bounds can be used as a test to ensure that the final value of T is reasonable. This paper presents theoretical lower and upper bounds for T for a given P_{fa}. In addition to being useful for the computation and validation of T, these results can be extended to derive good approximations to T.

This article is organized as follows. Section 2 introduces the FFT filter bank-based summation CFAR detector. Section 3 presents the formulas which relate T and P_{fa} for overlapped and non-overlapped signal data. Section 4 formulates lower and upper bounds for these two cases, while Section 5 concludes this paper.

2. THE FFT FILTER BANK-BASED L-BLOCK SUMMATION CFAR DETECTOR

Assume a band-limited signal that is uniformly sampled at a rate of F_s samples per second. Let there be M channels uniformly distributed across the frequency range contained within the Nyquist bandwidth. Assume that K FFT bins are assigned to each channel and that N FFT bins ($N \leq K$) centered within each channel are used to estimate the power contained within the channel. Consequently, an FFT of length MK is needed to compute the power levels for the M channels. Without any loss of generality, assume $K-N$ is an even integer. Let $\mathbf{w} = [w_0, \cdots, w_{MK-1}]^t$ be a linear phase FIR filter of length MK, where the superscript t denotes matrix (vector) transposition. Let $L \geq 1$ be any positive integer and consider L consecutive overlapping sample vectors $\mathbf{S}_l$ constructed as follows :

$$\mathbf{S}_l = [r_{l(1-\gamma)MK+MK-1}, \; \cdots, \; r_{l(1-\gamma)MK}]^t \tag{1}$$

$0 \leq l \leq L-1$. Here, r_n is the n-th sample of the input data stream, $0 \leq \gamma \leq \frac{1}{2}$ is the overlap ratio and γMK is required to be an integer. In practice, γ is often selected to be either 0 or $\frac{1}{2}$ and in the case of $\gamma = 0$, no data overlapping actually

1-4244-0469-X/06/$20.00

takes place. For each l, the two input vectors $\mathbf{S}_l$ and $\mathbf{S}_{l+1}$ have γMK samples in common. The vectors $\mathbf{S}_l$ are windowed by the windowing sequence $\mathbf{w}$, resulting in the windowed sample vectors $\mathbf{X}_l$:

$$\mathbf{X}_l = [w_0 r_{l(1-\gamma)MK+MK-1},\ \cdots,\ w_{MK-1} r_{l(1-\gamma)MK}]^t$$

The vectors $\mathbf{X}_l$ are then transformed by the inverse discrete Fourier transform matrix $\mathbf{F}$ of dimensions $MK \times MK$ to yield the FFT filter bank output sample vectors $\mathbf{Y}_l$:

$$\mathbf{Y}_l = \mathbf{F}\mathbf{X}_l = [y_{l,0},\ y_{l,1},\ \cdots,\ y_{l,MK-1}]^t$$

where

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi jl}{MK}} & \cdots & e^{\frac{2\pi j(MK-1)l}{MK}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi j(MK-1)}{MK}} & \cdots & e^{\frac{2\pi j(MK-1)(MK-1)}{MK}} \end{bmatrix} \quad (2)$$

From each vector $\mathbf{Y}_l$, a vector $\mathbf{z}_l = [z_{l,0}, \cdots, z_{l,M-1}]^t$ of length M is formed by summing the power from the N FFT bins centered within each channel:

$$z_{l,k} = \sum_{m=0}^{N-1} |y_{l,kK+\frac{K-N}{2}+m}|^2 , \quad 0 \le l \le L-1 ,\ 0 \le k \le M-1 . \quad (3)$$

In other words, the power from the N FFT bins with indices $I, I+1, \cdots, I+(N-1)$ is summed to form the power of the k-th channel for the data block $\mathbf{S}_l$, where $I = kK + \frac{K-N}{2}$. The detection criterion for the FFT filter bank-based L-block summation CFAR detector is defined as follows: For a given P_{fa} and corresponding threshold T, if $\sum_{l=0}^{L-1} z_{l,k} > T$, a signal is declared to exist in the k-th channel, otherwise, it is declared that there is no signal in the k-th channel. For brevity, the FFT filter bank-based L-block summation CFAR detector shall simply be called the L-block summation CFAR detector in this article.

3. THE PROBABILITY OF FALSE ALARM P_{fa} FOR THE L-BLOCK SUMMATION CFAR DETECTOR

Assume the input data stream r_n is a zero-mean complex-valued white Gaussian noise sequence with $E(r_p r_q^*) = \sigma^2 \delta_{p,q}$, where $\sigma^2 > 0$ is the noise variance (noise floor) and $\delta_{p,q} = 1$ if $p = q$ and $\delta_{p,q} = 0$ if $p \neq q$. For a given threshold, T, the corresponding probability of false alarm, P_{fa}, of the L-block summation CFAR detector is defined by

$$P_{fa} = \Pr\left\{ \sum_{l=0}^{L-1} z_{l,k} \ge T \right\} \quad (4)$$

The following theorems provide the theoretical basis for computing the threshold, T, for the L-block summation CFAR detector:

Theorem 1. (c.f. [1], [2], [6]) Assume $L \ge 2$ and $0 < \gamma \le \frac{1}{2}$. For a given threshold $T > 0$, the corresponding probability of false alarm P_{fa} for the L-block summation CFAR detector is given by:

$$P_{fa} = \sum_{m=1}^{LN} \frac{\lambda_m^{LN-1}}{\prod_{1 \le l \le LN, l \ne m} (\lambda_m - \lambda_l)} e^{-\frac{T}{\sigma^2 \lambda_m}} \quad (5)$$

where λ_m, $1 \le m \le LN$, are the LN distinct positive eigenvalues of the $L \times L$ block matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{B}^H & \mathbf{A} \end{bmatrix} \quad (6)$$

In (6), $\mathbf{H}$ is of dimensions $LN \times LN$, $\mathbf{0}$ is the $N \times N$ zero matrix and $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices defined respectively by:

$$\mathbf{A} = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1q} & \cdots & \tau_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{p1} & \tau_{p2} & \cdots & \tau_{pq} & \cdots & \tau_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{N1} & \tau_{N2} & \cdots & \tau_{Nq} & \cdots & \tau_{NN} \end{bmatrix} \quad (7)$$

$$\tau_{pq} = \sum_{l=0}^{MK-1} w_l^2 \exp \frac{2\pi jl(p-q)}{MK} \quad (8)$$

and

$$\mathbf{B} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1q} & \cdots & \gamma_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p1} & \gamma_{p2} & \cdots & \gamma_{pq} & \cdots & \gamma_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{Nq} & \cdots & \gamma_{NN} \end{bmatrix} \quad (9)$$

$$\gamma_{pq} = e^{-2\pi j(1-\gamma)(I+q-1)} \times \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK} \exp \frac{2\pi jl(p-q)}{MK} \quad (10)$$

Let

$$\beta_m = \frac{\lambda_m^{LN-1}}{\prod_{1 \le l \le LN, l \ne m} (\lambda_m - \lambda_l)} ,\ 1 \le m \le LN . \quad (11)$$

It can be shown that

$$\sum_{m=1}^{LN} \beta_m = 1 \quad (12)$$

The sequence β_m alternates in sign and can sometimes fluctuate quite erratically when the spacings between some of the eigenvalues λ_m, $1 \le m \le LN$, are very small. This significantly contributes to the numerical difficulties associated with the computation of T.

Theorem 2. (c.f. [2], [6]) Assume $L \ge 1$, $N > 1$ and $\gamma = 0$. For a given threshold $T > 0$, the probability of false alarm P_{fa} for the L-block summation CFAR detector is given by

$$P_{fa} = \sum_{m=1}^{N} \sum_{k=1}^{L} A_{mk} \sum_{t=0}^{k-1} \frac{\left(\frac{T}{\sigma^2 \mu_m}\right)^t}{t!} e^{-\frac{T}{\sigma^2 \mu_m}} \tag{13}$$

where the coefficients A_{mk}, $1 \le m \le N$, $1 \le k \le L$, are defined by

$$\begin{aligned} A_{mL} &= \left(\frac{\mu_m^{N-1}}{\prod_{1 \le l \le N, l \ne m} (\mu_m - \mu_l)} \right)^L, \\ A_{mk} &= A_{mL} \times \sum_{k_1 + \cdots + k_{m-1} + k_{m+1} + \cdots + k_N = L-k} \\ &\quad \Gamma_m(k_1, k_2, \cdots, k_{m-1}, k_{m+1}, \cdots, k_N) \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\Gamma_m(k_1, k_2, \cdots, k_{m-1}, k_{m+1}, \cdots, k_N) \\ = &\prod_{1 \le l \le N, l \ne m} \frac{(L + k_l - 1)!}{k_l!\,(L-1)!} \left(\frac{\mu_l}{\mu_l - \mu_m} \right)^{k_l} \end{aligned}$$

Note that k_l, $1 \le l \le N$, are non-negative integers and μ_l, $1 \le l \le N$, are the N distinct positive eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (7). It can be verified that

$$\sum_{m=1}^{N} \sum_{k=1}^{L} A_{mk} = 1 \tag{15}$$

The sequences A_{mk} and β_m (defined in (11)) behave in a similar way.

Theorem 3. (c.f. [2], [6]) Let $L \ge 1$, $N = 1$ and $\gamma = 0$. For a given threshold $T > 0$, the probability of false alarm P_{fa} for the L-block summation CFAR detector is given by

$$P_{fa} = \sum_{t=0}^{L-1} \frac{\left(\frac{T}{\lambda}\right)^t}{t!} e^{-\frac{T}{\lambda}} \tag{16}$$

where

$$\lambda = \sigma^2 \sum_{l=0}^{MK-1} w_l^2 \tag{17}$$

4. LOWER AND UPPER BOUNDS

In this section, we present several lower and upper bounds for T for overlapped and non-overlapped signal data. The technical derivations of these results, omitted due to space constraints, will appear in a forthcoming publication.

4.1. Overlapped Input Data

Theorem 4 . Let $L \ge 2$ and $0 < \gamma \le \frac{1}{2}$. Let

$$\lambda_{max} = \max_{1 \le l \le LN} \{\lambda_l\} \tag{18}$$

be the maximum eigenvalue of the positive definite Hermitian matrix $\mathbf{H}$ defined by (6). Then

$$-\lambda_{max} \sigma^2 \ln P_{fa} \le T \tag{19}$$

The lower bound given in **Theorem 4** is easily computed. However, better lower bounds can be obtained, as demonstrated by the following theorem:

Theorem 5 . Let $L \ge 2$, $0 < \gamma \le \frac{1}{2}$ and assume $\lambda_1 > \lambda_2 > \cdots > \lambda_{LN}$ where λ_m, $1 \le m \le LN$, are the LN positive eigenvalues of the $LN \times LN$ Hermitian matrix $\mathbf{H}$ defined by (6). For any integer k, $2 \le k \le LN$, let the solution for $\overline{T}$ of the following equation be denoted by $T(P_{fa}, k)$:

$$P_{fa} = \sum_{m=1}^{k} \frac{\lambda_m^{k-1}}{\prod_{1 \le l \le k, l \ne m} (\lambda_m - \lambda_l)} e^{-\frac{\overline{T}}{\sigma^2 \lambda_m}} \tag{20}$$

Then

$$\begin{aligned} &-\lambda_{max} \sigma^2 \ln P_{fa} \le T(P_{fa}, 2) \le \cdots \le T(P_{fa}, k) \le \\ &T(P_{fa}, k+1) \le \cdots \le T(P_{fa}, LN) = T \end{aligned} \tag{21}$$

For small values of k, the lower bounds $T(P_{fa}, k)$ are relatively easy to compute numerically as the eigenvalues used in (20), namely, $\lambda_1 > \lambda_2 > \cdots > \lambda_k$, are reasonably large and widely spaced.

Theorem 6 . Let $L \ge 2$, $0 < \gamma \le \frac{1}{2}$ and define

$$P_0 = \left(\sum_{m=1}^{LN} |\beta_m| \right)^{-\frac{\lambda_{max}}{LN \sum_{l=0}^{MK-1} w_l^2 - \lambda_{max}}} \tag{22}$$

where β_m and λ_{max} are defined by (11) and (18) respectively. If $0 < P_{fa} < P_0$, then

$$T \le -LN\sigma^2 (\ln P_{fa}) \sum_{l=0}^{MK-1} w_l^2 \tag{23}$$

The upper bound given in (23) is a universal upper bound in the sense that it depends only on L, N and P_{fa} for a normalized window (that is, for windows satisfying the constraint

$\sum_{l=0}^{MK-1} w_l^2 = 1$). Preliminary tests indicate that for relatively small P_{fa}, the inequality (23) holds for many windows.

The following theorem may provide a better upper bound.

Theorem 7. Let $L \geq 2$ and $0 < \gamma \leq \frac{1}{2}$. For any positive integer $m \geq 1$, let the unique solution for z of the following equation be denoted by $T_m(P_{fa})$:

$$e^{-z}\left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^{m-1}}{(m-1)!}\right) = P_{fa} \qquad (24)$$

Then

$$\begin{aligned} T &\leq \lambda_{max}\sigma^2 T_{LN}(P_{fa}) & (25)\\ &\leq \lambda_{max}\sigma^2\left[LN - 1 - 3\sqrt{(LN-1)}\ \ln P_{fa}\right] \end{aligned}$$

where λ_{max} is defined by (18).

4.2. Non-Overlapped Input Data

Theorem 8 . Let $L \geq 1$, $N \geq 2$ and $\gamma = 0$. Let

$$\mu_{max} = \max_{1 \leq l \leq N}\{\mu_l\} \qquad (26)$$

be the maximum eigenvalue of the Hermitian matrix $\mathbf{A}$ defined by (7). Then

$$-\mu_{max}\sigma^2 \ln P_{fa} \leq \mu_{max}\sigma^2 T_L(P_{fa}) \leq T \qquad (27)$$

Theorem 9 . Let $L \geq 1$, $N \geq 2$ and $\gamma = 0$. Let μ_{max} be defined by (26). Then

$$\begin{aligned} T &\leq \mu_{max}\sigma^2 T_{LN}(P_{fa}) & (28)\\ &\leq \mu_{max}\sigma^2\left[LN - 1 - 3\sqrt{(LN-1)}\ \ln P_{fa}\right] \end{aligned}$$

Theorem 10 . Let $L \geq 1$, $N \geq 2$ and $\gamma = 0$. Let μ_l, $1 \leq l \leq N$, be the N distinct positive eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (7). Let T_1 and T_2 be the solutions for $\overline{T}$ of the equations (29) and (30) respectively:

$$\sum_{m=1}^{N} A_m e^{-\frac{\overline{T}}{\sigma^2\mu_m}} = (P_{fa})^{\frac{1}{L}} \qquad (29)$$

$$\sum_{m=1}^{N} A_m\, e^{-\frac{\overline{T}}{\sigma^2\mu_m}} = 1 - (1 - P_{fa})^{\frac{1}{L}} \qquad (30)$$

where $A_m = \frac{\mu_m^{N-1}}{\prod_{1 \leq l \leq N, l \neq m}(\mu_m - \mu_l)}$. We have

$$LT_1 \leq T \leq LT_2 \qquad (31)$$

5. CONCLUSIONS

Lower and upper bounds for the threshold T of the FFT filter bank-based summation CFAR detector have been derived for overlapped and non-overlapped signal data. These bounds are useful for initializing the computation of T for a given probability of false alarm P_{fa}. The formula (5) is sensitive to rounding errors while the formula (13) is more robust, though both may fail for large N or L. We plan to extend these results to derive tighter bounds for T, in the process, obtaining a good approximation for T.

6. ACKNOWLEDGEMENT

We would like to thank our colleague Dr. Sreeraman Rajan for stimulating discussions.

7. REFERENCES

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