Approaching Near Optimal Detection Performance via Stochastic Resonance

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***Abstract*— This paper considers the stochastic resonance (SR) effect in the two hypotheses signal detection problem. Performance of a SR enhanced detector is derived in terms of the probability of detection P_D and the probability of false alarm P_{FA}. Furthermore, the conditions required for potential performance improvement using SR are developed. Expression for the optimal stochastic resonance noise pdf which renders the maximum P_D without increasing P_{FA} is derived. By further strengthening the conditions, this approach yields the constant false alarm rate (CFAR) receiver. Finally, detector performance comparisons are made between the optimal SR noise, Gaussian, Uniform and optimal symmetric pdf noises.**

I. Introduction

Stochastic resonance (SR) is a nonlinear physical phenomenon in which the output signals of some nonlinear systems can be amplified by adding noise. Since its discovery by Benzi et al. in 1981 [1], the SR effect has been observed and applied in numerous nonlinear systems [2]. The classic SR signature is the signal-to-noise ratio (SNR) gain of certain systems, i.e, in some nonlinear systems, the output SNR is significantly higher than the input SNR when an appropriate amount of noise is added [2], [3]. In signal detection theory, SR also plays a very important role to improve signal detectability. In [4] and [5], improvement of detection performance of a weak sinusoid signal is reported. To detect a DC signal in a Gaussian mixture noise background, Kay [6] showed that under certain conditions, performance of the sign detector can be enhanced by adding some white Gaussian noise. A study of the phenomenon of stochastic resonance in quantizers showed that a better detection performance can be achieved by a proper choice of the quantizer thresholds [7]. Recently, it was pointed out that the detection performance can be further improved by using an optimal detector on the output signal [8]. Despite the progress achieved by the above approaches, SR effects are only reported in a very limited number of signal detection systems. In this paper, we try to explore the underlying mechanism of this SR phenomenon for a more general two hypotheses detection problem.

Consider a two hypotheses detection problem where given a N dimensional data vector $\mathbf{x} \in R^N$, we have to decide between two hypotheses H_1 or H_0,

$$\begin{cases} H_0\colon p_{\mathbf{x}}(\mathbf{x}; H_0) = p_0(\mathbf{x}) \\ H_1\colon p_{\mathbf{x}}(\mathbf{x}; H_1) = p_1(\mathbf{x}) \end{cases}, \tag{1}$$

where $p_0(\mathbf{x})$ and $p_1(\mathbf{x})$ are the pdfs of $\mathbf{x}$ under H_0 and H_1, respectively. In order to make a decision, a test (possibly randomized) is needed to choose between the two hypotheses. This test can be completely characterized by a *critical function* (*decision function*) ϕ where $0 \leq \phi(\mathbf{x}) \leq 1$ for all $\mathbf{x}$. For any observation $\mathbf{x}$, this test chooses the H_1 hypothesis with probability $\phi(\mathbf{x})$. In many cases, $\phi(\mathbf{x})$ can be implicitly expressed by using a test statistic T which is a function of $\mathbf{x}$ and a threshold η such that

$$T(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta. \tag{2}$$

Therefore, probability of detection P_D is given by

$$P_D^x = \int \phi(\mathbf{x}) p_1(\mathbf{x}) d\mathbf{x}, \tag{3}$$

and the probability of false alarm P_{FA} is given by

$$P_{FA}^x = \int \phi(\mathbf{x}) p_0(\mathbf{x}) d\mathbf{x}, \tag{4}$$

where the superscripts on P_D and P_{FA} in (3) and (4) denote that the test in (2) is employed for the data vector $\mathbf{x}$. Although a Neyman-Pearson detector is optimum in the sense of maximizing P_D given a fixed P_{FA}, it requires the complete knowledge of the pdfs $p_0(\cdot)$ and $p_1(\cdot)$ which are not always available in a practical application and sometime too complicated to be implemented. Therefore, some suboptimal detectors that are simpler and more robust are used in numerous applications. For some suboptimal detectors, as Kay pointed out in [6], detection performance can be improved by adding an independent noise to the data under certain conditions. However, the underlying mechanism of this SR phenomenon has not been fully explored. Furthermore, it raises the more interesting question as to how we determine the best 'noise' to be added in order to achieve the best achievable detection performance for the suboptimal detector. In this case, the detection problem can be stated as: Given that the test is fixed; i.e., the critical function $\phi(\cdot)$ (e.g., T and η) is fixed, what kind of noise and how much noise (i.e., noise pdf) should we add to the observed data to maximize P_D without increasing P_{FA}? In this paper, a theoretical analysis is presented to gain further insight into the SR phenomenon, and the detection performance of the noise enhanced observations is obtained. Furthermore, the optimum noise pdf, i.e., not only the noise level but also the noise types is determined.

II. Noise Enhanced Detection

In order to enhance detection performance, we add noise to the original data process $\mathbf{x}$ and obtain a new data process $\mathbf{y}$ given by

$$\mathbf{y} = \mathbf{x} + \mathbf{n}, \tag{5}$$

where $\mathbf{n} \in R^N$ is either an independent random process with pdf $p_{\mathbf{n}}(\cdot)$ or a nonrandom signal. Notice that here we do not have any constraint for $\mathbf{n}$. For example, $\mathbf{n}$ can even be a deterministic signal $A \in R^N$, corresponding to $p_{\mathbf{n}}(\mathbf{n}) = \delta(\mathbf{n} - A)$. The pdf of $\mathbf{y}$ is expressed by the convolution of the pdfs such that

$$p_{\mathbf{y}}(\mathbf{y}) = p_{\mathbf{x}}(\mathbf{x}) * p_{\mathbf{n}}(\mathbf{x}) = \int p_{\mathbf{x}}(\mathbf{x}) p_{\mathbf{n}}(\mathbf{y} - \mathbf{x}) d\mathbf{x}. \tag{6}$$

The binary hypotheses testing problem for this new observed data $\mathbf{y}$ can be expressed as:

$$\begin{cases} H_0\text{: } p_{\mathbf{y}}(\mathbf{y}; H_0) = \int p_0(\mathbf{x}) p_{\mathbf{n}}(\mathbf{y} - \mathbf{x}) d\mathbf{x} \\ H_1\text{: } p_{\mathbf{y}}(\mathbf{y}; H_1) = \int p_1(\mathbf{x}) p_{\mathbf{n}}(\mathbf{y} - \mathbf{x}) d\mathbf{x} \end{cases}. \tag{7}$$

Since the detector is fixed, i.e., the critical function ϕ of $\mathbf{y}$ is precisely the one used for $\mathbf{x}$, the P_D based on data $\mathbf{y}$ is given by,

$$\begin{aligned} P_D^y &= \int \phi(\mathbf{y}) p_{\mathbf{y}}(\mathbf{y}; H_1) d\mathbf{y} \\ &= \int p_1(\mathbf{x}) C_{\mathbf{n},\phi}(\mathbf{x}) d\mathbf{x}, \end{aligned} \tag{8}$$

where

$$C_{\mathbf{n},\phi}(\mathbf{x}) \equiv \int \phi(\mathbf{y}) p_{\mathbf{n}}(\mathbf{y} - \mathbf{x}) d\mathbf{y}. \tag{9}$$

Alternatively,

$$\begin{aligned} P_D^y &= \int p_{\mathbf{n}}(\mathbf{x}) \left(\int \phi(\mathbf{y}) p_1(\mathbf{y} - \mathbf{x}) d\mathbf{y} \right) d\mathbf{x} \\ &= \int F_{1,\phi}(\mathbf{x}) p_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}, \end{aligned} \tag{10}$$

where

$$F_{i,\phi}(\mathbf{x}) \equiv \int \phi(\mathbf{y}) p_i(\mathbf{y} - \mathbf{x}) d\mathbf{y}, \tag{11}$$

$i = 0, 1$ corresponding to hypothesis H_i and $P_D^x = F_{1,\phi}(0)$. Similarly, we have,

$$P_{FA}^y = \int p_0(\mathbf{x}) C_{\mathbf{n},\phi}(\mathbf{x}) d\mathbf{x} \tag{12}$$

$$= \int F_{0,\phi}(\mathbf{x}) p_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}, \tag{13}$$

and $P_{FA}^x = F_{0,\phi}(0)$. To simplify notation, we omit the subscript ϕ of F and C and denote them as F_1, F_0 and C_n, respectively. Further, from (11), $F_1(\mathbf{x}_0)$ and $F_0(\mathbf{x}_0)$ are actually the conditional P_D and P_{FA} for this detection scheme with input $\mathbf{y} = \mathbf{x} + \mathbf{x}_0$, respectively. From (10) and (13), we may formalize the definition of the optimal SR noise as follows.

Consider the two hypotheses detection problem as in (1). The pdf of optimum SR noise is given by

$$p_{\mathbf{n}}^{opt} = \arg\max_{p_{\mathbf{n}}} \int F_1(\mathbf{x}) p_{\mathbf{n}}(\mathbf{x}) d\mathbf{x} \tag{14}$$

where

1) $p_{\mathbf{n}}(\mathbf{x}) \geq 0, \ \mathbf{x} \in R^N$.
2) $\int p_{\mathbf{n}}(\mathbf{x}) d\mathbf{x} = 1$.
3) $\int F_0(\mathbf{x}) p_{\mathbf{n}}(\mathbf{x}) d\mathbf{x} \leq F_0(0)$.

Conditions 1) and 2) are fundamental properties of a pdf function. Condition 3) ensures that $P_{FA}^y \leq P_{FA}^x$, i.e., the P_{FA} constraint specified under the Neyman-Pearson Criterion is satisfied. Further, if the inequality of condition 3) becomes equality[1], this detector is CFAR.

III. Optimum SR Noise

Let us consider the relationship between $p_{\mathbf{n}}(\mathbf{x})$ and $F(\mathbf{x})$. By (11), for a given value f_0 of F_0, we have $\mathbf{x} = F_0^{-1}(f_0)$, where F_0^{-1} is the inverse function of F_0. When F_0 is a one-to-one mapping function, $\mathbf{x}$ only takes one value. Otherwise, $F_0^{-1}(f_0)$ is a set of $\mathbf{x}$ where $F_0(\mathbf{x}) = f_0$. Therefore, we have a function/mapping relationship between F_1 and F_0 given by

$$f_1 = F_1(F_0^{-1}(f_0)). \tag{15}$$

Furthermore, the conditions on the optimum noise can be rewritten in terms of f_0 equivalently as

4) $p_{\mathbf{n},f_0}(f_0) \geq 0$.
5) $\int p_{\mathbf{n},f_0}(f_0) df_0 = 1$.
6) $\int f_0 p_{\mathbf{n},f_0}(f_0) df_0 \leq P_{FA}^x$.

and

$$P_D^y = \int f_1 p_{\mathbf{n},f_0}(f_0) df_0 \tag{16}$$

Before determining the exact form of $p_{\mathbf{n}}^{opt}$, we first state the following theorem for the form of optimum SR noise.

*Theorem 1 (**Form of Optimum SR Noise**):* To maximize P_D^y, under the constraint that $P_{FA}^y \leq P_{FA}^x$, the optimum noise can be assumed to take the following form [2]

$$p_{\mathbf{n}}^{opt}(\mathbf{n}) = \lambda \delta(\mathbf{n} - \mathbf{n}_1) + (1 - \lambda) \delta(\mathbf{n} - \mathbf{n}_2) \tag{17}$$

where $0 \leq \lambda \leq 1$. In other words, to obtain the maximum achievable detection performance, the optimum noise is a randomization of two discrete DC vectors added with probability λ and $1 - \lambda$, respectively.

From Theorem 1, with $f_{0i} = F_0(\mathbf{n}_i)$ and $\mathbf{n}_i$ such that $F_1(\mathbf{n}_i) = f_{1i} = F_{1,max}(f_0 i)$, $i = 1, 2$, we have

$$P_{D,opt}^y = \lambda F_{1,max}(f_{01}) + (1 - \lambda) F_{1,max}(f_{02}), \tag{18}$$

and

$$P_{FA,opt}^y = \lambda f_{01} + (1 - \lambda) f_{02} \leq P_{FA}^x. \tag{19}$$

The optimum SR noise can also be expressed in terms of $C_{\mathbf{n}}$, such that

$$C_{\mathbf{n}}^{opt}(\mathbf{n}) = \lambda \phi(\mathbf{n} + \mathbf{n}_0) + (1 - \lambda) \phi(\mathbf{n} + \mathbf{n}_1). \tag{20}$$

[1] In many cases, the optimal P_D is achieved when the equality holds.

[2] This form of optimum noise pdf has not been proven to be unique. There may exist other forms of noise pdf that achieve the same detection performance.

Let $F_{1,max}(t) = \sup(f_1 : (f_1, f_0) \in U, f_0 = t)$.For any noise $p_{\mathbf{n}}$, we have

$$P_D^y(p_{\mathbf{n}}) \leq \int F_{1,max}(f_0) p_{\mathbf{n},f_0}(f_0) df_0. \tag{21}$$

From Theorem 1, some useful conclusions can be drawn.

Lemma 1: Let $F_{1M} = \max(F_{1,max}(t))$. If there exists a t_o such that $t_o < F_0(0)$ and $F_{1,max}(t_o) = F_{1M}$, the optimum noise pdf with $F_0(\mathbf{n}_o) = n_o$ and $F_1(\mathbf{n}_o) = F_{1M}$ can be written as

$$P_{\mathbf{n}}^{opt}(\mathbf{n}) = \delta(\mathbf{n} - \mathbf{n}_o), \tag{22}$$

where the noise is now deterministic.

Lemma 2: If $F_{1,max}(f_0)$ is continuously differentiable, then the line connecting (f_{11}, f_{01}) and (f_{12}, f_{02}) is the co-tangent line of $F_{1,max}(\cdot)$. Furthermore, if $F_{1,max}(f_0)$ is monotonically increasing, $P_{FA,opt}^y = P_{FA}^x$.

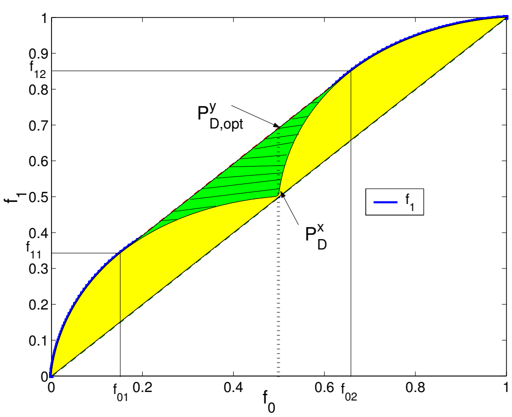

Fig. 1. Relationship between f_1 and f_0. The green dashed region is the region where a possible SR effect may take place and provides the conditions required for potential performance improvement.

Depending on the specific properties of $F_{1,max}$, we may also determine the improvability of this detector by adding SR noise. The sufficient conditions of improvability and non-improvability are given in the following theorems.

*Theorem 2 (**Improvability of Detection via SR**):* Suppose $F_{1,max}$ is second-order continuously differentiable around $F_0(0)$. If $F''_{1,max}(F_0(0)) > 0$, then there exists at least one noise process $\mathbf{n}$ with pdf $p_{\mathbf{n}}(\cdot)$ that can improve the detection performance.

*Theorem 3 (**Non-improvability of Detection via SR**):* If there exists a monotonic, non-decreasing concave function $\Psi(f_0)$ where $\Psi(F_0(0)) = F_{1,max}(F_0(0)) = F_1(0)$ and $\Psi(f_0) \geq F_{1,max}(f_0)$ for every f_0, then $P_D^y \leq P_D^x$ for any independent noise, i.e., the detection performance can not be improved by adding noise.

IV. A Detection Example

Here, we consider the same detection problem as considered by Kay [6].

$$\begin{cases} \text{H}_0\text{:}x[i] = w[i] \\ \text{H}_1\text{:}x[i] = A + w[i] \end{cases}, \tag{23}$$

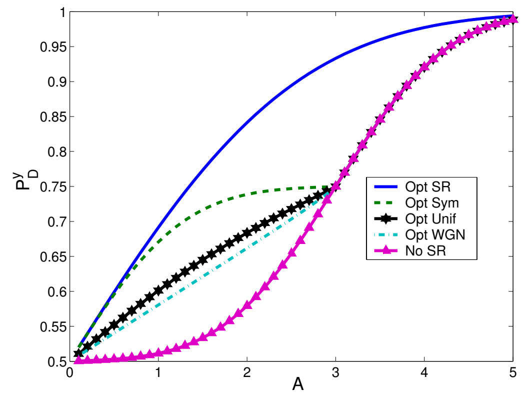

Fig. 2. P_D^y as a function of signal level A in Gaussian mixture noise ($\mu = 3$ and $\sigma_0 = 1$). When $A \geq \mu$, adding any non zero symmetric noise will only decrease the detection performance. Therefore, the detection performance of “opt sym”, “opt Unif”, “opt WGN” and “NO SR” are the same when $A \geq \mu$.

for $i = 0, 1, \cdots, N-1$, $A > 0$ is a known dc signal, and $w[i]$ is i.i.d symmetric Gaussian mixture noise pdf

$$p_w(w) = \frac{1}{2}\gamma(w; -\mu, \sigma_0^2) + \frac{1}{2}\gamma(w; \mu, \sigma_0^2), \tag{24}$$

where $\gamma(w; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(w-\mu)^2}{2\sigma^2}\right]$, $\mu = 3$, $A = 1$ and $\sigma_0 = 1$. A suboptimal detector is considered with test statistic

$$T(x) = \frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{1}{2} + \frac{1}{2}\text{sgn}(x[i])\right) = \frac{1}{N}\sum_{i=0}^{N-1}\varpi_x[i], \tag{25}$$

where $\varpi_x[i] = \frac{1}{2} + \frac{1}{2}\text{sgn}(x[i])$. From (25), this detector is essentially a fusion of the decision results of N i.i.d. sign detectors ($N = 1$). When $N = 1$, comparing the model (1) and the model (23), for each detector, we have test statistic $T_1(x) = x$, threshold $\eta = 0$ and $P_{FA}^x = 0.5$. The distribution of x under the H_0 and H_1 hypotheses can be expressed as

$$p_0(x) = \frac{1}{2}\gamma(x; -\mu, \sigma_0^2) + \frac{1}{2}\gamma(x; \mu, \sigma_0^2), \tag{26}$$

$$p_1(x) = \frac{1}{2}\gamma(x; -\mu + A, \sigma_0^2) + \frac{1}{2}\gamma(x; \mu + A, \sigma_0^2), \tag{27}$$

respectively. The critical function is given by

$$\phi(x) = 1 \ x > 0, \ \ 0 \ x \leq 0.$$

The problem of determining the optimal SR noise is to find the optimal $p(n)$ where for the new observation $y = x + n$, the $P_D^y = p(y > 0; H_1)$ is maximum while the $P_{FA}^y = p(y > 0; H_0) \leq P_{FA}^x = \frac{1}{2}$. When $N > 1$, it can be shown that the

TABLE I
COMPARISON OF DIFFERENT DETECTION PERFORMANCE FOR DIFFERENT SR NOISE PDF.

SR	p_n^{opt}	p_s^{opt}	p_u^{opt}	p_g^{opt}	No SR
P_D^y	0.6915	0.6707	0.6011	0.5807	0.5114

detection performance is monotonically increasing with P_D^y when the probability of false alarm $P_{FA}^y = \frac{1}{2}$. Therefore, as we restrict the additive noise n to be an i.i.d noise, the optimal noise of $N > 1$ is the same as $N = 1$ for each data sample. In the following discussion, we only consider the one sample case ($N = 1$). The performance of the $N > 1$ case can be derived similarly. From (10) and (13), it can be shown that in this case, $F_1(x) = \frac{1}{2}Q(\frac{-x-\mu-A}{\sigma_0}) + \frac{1}{2}Q(\frac{-x+\mu-A}{\sigma_0})$ and $F_0(x) = \frac{1}{2}Q(\frac{-x-\mu}{\sigma_0}) + \frac{1}{2}Q(\frac{-x+\mu}{\sigma_0})$, where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-t^2}{2}} dt$. It also follows that $f_1 > f_0$ is monotonically increasing with f_0. Therefore, $F_{1,max}(f_0) = f_1$. From Lemma (2), it can be shown that $p_n^{opt}(n) = 0.3085\delta(n+3.5) + 0.6915\delta(n-2.5)$, and $P_{D,opt}^y = 0.6915$. We also determine the optimal SR noise parameters for three different types of noise pdf, namely symmetric noise with arbitrary pdf $p_s(x)$, white Gaussian noise $p_g(x) = \gamma(x; 0, \sigma^2)$ and white uniform noise $p_u(x) = \frac{1}{a}, a > 0, , -\frac{a}{2} \le x \le \frac{a}{2}$. The noise altered data process are denoted as y_s, y_u and y_g, respectively. For the arbitrary symmetrical noise case, given that $A < \mu$, $\sigma_0 < \sigma_1$ and μ large enough ($2\mu \pm A > 3\sigma_0$), we have $p_s^{opt} = \frac{1}{2}\delta(x-\mu) + \frac{1}{2}\delta(x+\mu)$ and $P_{D,opt}^{y_s} = \frac{1}{2}Q\left(-\frac{A}{\sigma_0}\right) + \frac{1}{4} = 0.6707$, the maximum value of $P_D^{y_s}$ is not dependent on μ. Similarly, for the uniform noise cases, $a_{opt} = 8.4143$; for the Gaussian noise case, $\sigma_{opt}^2 = 7.6562$. Table I shows the different $P_{D,opt}^y$ for these different types of SR noise. Fig. 1 shows the relationship between f_1 and f_0. The green dashed region is the region of (f_1, f_0) where a possible SR effect may take place and provides the conditions required for potential performance improvement. Fig. 2 shows P_D^x as well as the maximum achievable P_D^y with different values of A. The detection performance is significantly improved by adding optimal SR noise. The maximum detection performance of different SR noise enhanced detectors w.r.t σ_0 are shown in Fig. 3. Fig 4 shows the ROC curve for this detection problem when $N = 30$. As expected, the optimum SR detector provides superior detection performance.

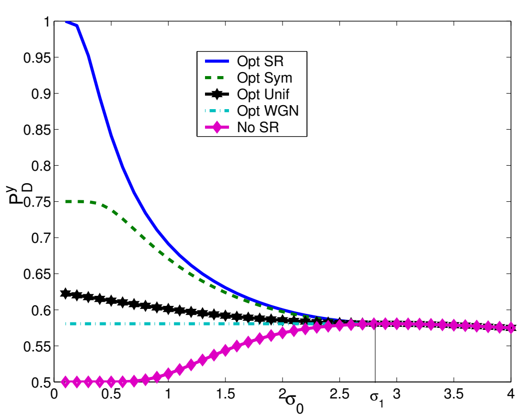

Fig. 3. P_D^y as a function of σ_0 for different types of noise when $\mu = 3$ and $A = 1$. The improvement of P_D monotonically decreases when σ_0 increases. When $\sigma_0 > \sigma_1$, the detection performance can not be improved by adding SR noise.

V. Concluding Remarks

In this paper, we have outlined the fundamental mechanism responsible for enhanced SR detection. The exact form of the optimal SR noise pdf has been proposed. Further, we establish the conditions of potential improvement of P_D via the SR effect.The optimal SR noise is shown to be a proper randomization of no more than two discrete dc signals. For some suboptimal detectors, we show that under some conditions, adding an appropriate noise may improve its detection performance. By adding the optimal SR noise to the observed data process, a significant improvement of P_D is reported. Details are found in [9], [10].

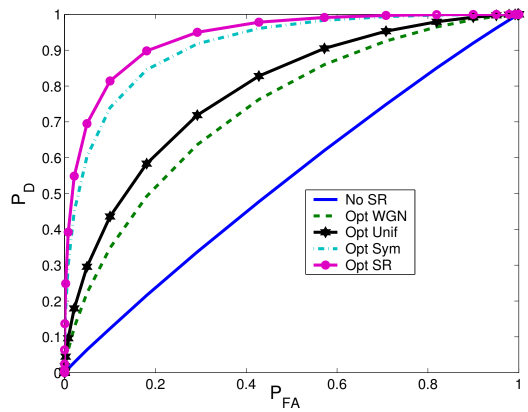

Fig. 4. ROC curves for different SR enhanced sign detections, N = 30.

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