GLRT-BASED DIRECTION DETECTORS IN NOISE AND SUBSPACE INTERFERENCE

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ABSTRACT

In this paper we propose decision schemes to distinguish between the H_0 hypothesis that range cells under test contain disturbance only (i.e., noise plus interference) and the H_1 hypothesis that they also contain signal components along a direction which is a priori unknown, but constrained to belong to a given subspace $\langle \boldsymbol{H} \rangle$ of the observables. The disturbance is modeled in terms of complex normal noise vectors plus deterministic interference assumed to belong to a known subspace $\langle \boldsymbol{J} \rangle$ of the observables. At the design stage we resort to either the plain Generalized Likelihood Ratio Test (GLRT) or the two-step GLRT-based design procedure. Moreover, we assume that a set of noise only (secondary) data is available. A preliminary performance analysis, conducted by resorting to simulated data, shows that the one-step GLRT performs better than the two-step GLRT-based design procedure.

1. INTRODUCTION

A High-Resolution Radar (HRR) can resolve a target into a number of scattering centers depending on the range extent of the target, the range resolution capabilities of the radar, and its operating frequency. Measurements indicate that radar properties of several targets can be modeled in terms of a set of scattering centers each parameterized by its range, amplitude and, possibly, polarization ellipse.

Properly designed HRR's allow significant improvements in terms of detection performance as shown in [1], [2], [3] and references therein. Those papers address adaptive radar detection of distributed targets embedded in possibly non-Gaussian disturbance; returns from target's scattering centers are modeled as signals known up to multiplicative deterministic factors. All of those papers rely on the assumption that a set of secondary data, free of signal components, but sharing the spectral properties of the data under test, is available. The case of point-like targets assumed to belong to a known subspace of the observables had been addressed in [4]. Finally, several detection algorithms for point-like or extended targets in Gaussian noise are encompassed as special cases of the amazingly general framework and derivation described in [5].

In this paper we attack adaptive detection of distributed targets embedded in homogeneous Gaussian noise with unknown covariance matrix plus interference; interference subspace is known and linearly independent of the signal space. The possible useful signals are aligned with an unknown direction constrained to belong to a given subspace of the observables. This model might be a viable means to address adaptive detection in case of mismatched steering vectors. It has been firstly proposed in [6] and [7] where detection in presence of white noise with known and unknown power, respectively, has been considered. Herein, we resort to a plain GLRT and to a two-step GLRT-based procedure, a point better specified in the following.

The paper is organized as follows: next section is devoted to the problem formulation while the detector design is the object of Section 3. Section 4 contains the performance assessment of the proposed algorithms and, finally, Section 5 concludes the paper with some remarks and hints for future work.

2. PROBLEM FORMULATION

Assume that an array of N_a antennas senses K_P range cells and that each antenna collects N_t samples from each of those cells. Denote by $\boldsymbol{r}_k$, $k \in \Omega_P \equiv \{1, \ldots, K_P\}$, the N-dimensional vector, with $N = N_a N_t$, containing returns from the k-th cell. We assume that the disturbance is the sum of colored noise and interference, modeled as a deterministic signal; we want to decide between the H_0 hypothesis that $\boldsymbol{r}_k$, $k \in \Omega_P$, contain disturbance only against the H_1 hypothesis that they also contain useful target echoes $\boldsymbol{s}_k$.

Moreover, we suppose that the $\boldsymbol{s}_k$'s can be modeled as $\boldsymbol{s}_k = \alpha_k \boldsymbol{s} \in \mathbb{C}^{N\times 1}$ with $\boldsymbol{s}$ being, in turn, a linear combination of r linearly independent modes; in addition, the interference signals $\boldsymbol{i}_k$, $k \in \Omega_P$, are linear combinations of q, $q + r \leq N$, linearly independent modes. Otherwise stated, $\boldsymbol{s}$ and $\boldsymbol{i}_k$, $k \in \Omega_P$, are assumed to belong to a r-dimensional subspace $\langle \boldsymbol{H} \rangle$ and a q-dimensional subspace $\langle \boldsymbol{J} \rangle$, respectively. Thus, $\boldsymbol{s}$ and $\boldsymbol{i}_k$ can be recast as $\boldsymbol{s} = \boldsymbol{H}\boldsymbol{p}$ and $\boldsymbol{i}_k = \boldsymbol{J}\boldsymbol{q}_k$, $k \in \Omega_P$, where $\boldsymbol{p}$ and $\boldsymbol{q}_k$, $k \in \Omega_P$, are r-dimensional and q-dimensional complex vectors, respectively.

In the following we assume that the subspaces spanned by the columns of the matrices $\boldsymbol{H}$ and $\boldsymbol{J}$ are known and that the matrix $[\boldsymbol{H}\ \boldsymbol{J}]$ is full rank.

The noise vectors $\boldsymbol{n}_k$'s, $k \in \Omega_P$, are modeled as N-dimensional complex normal vectors with unknown covariance matrix $\boldsymbol{M}$. We also suppose that K_S secondary data, $\boldsymbol{r}_k$, $k \in \Omega_S \equiv \{K_P + 1, \ldots, K_P + K_S\}$, containing noise only, are available and that the returns share the same statistical characterization of the noise components in the primary data. Finally, we assume that the $\boldsymbol{n}_k$'s, $k \in \Omega_P \cup \Omega_S$, are independent random vectors.

Summarizing, the detection problem to be solved can be formulated in terms of the following binary hypothesis test

$$\begin{cases} H_0 : \begin{cases} \boldsymbol{r}_k = \boldsymbol{J}\boldsymbol{q}_k + \boldsymbol{n}_k, & k \in \Omega_P, \\ \boldsymbol{r}_k = \boldsymbol{n}_k, & k \in \Omega_S, \end{cases} \\ \\ H_1 : \begin{cases} \boldsymbol{r}_k = \alpha_k \boldsymbol{H}\boldsymbol{p} + \boldsymbol{J}\boldsymbol{q}_k + \boldsymbol{n}_k, & k \in \Omega_P, \\ \boldsymbol{r}_k = \boldsymbol{n}_k, & k \in \Omega_S, \end{cases} \end{cases} \tag{1}$$

where we suppose that $K_S \geq N$ and, as already stated, that $r + q \leq N$.

3. DETECTOR DESIGN

Denote by $\boldsymbol{R} = [\boldsymbol{R}_P\ \boldsymbol{R}_S]$ the overall data matrix, where $\boldsymbol{R}_P = [\boldsymbol{r}_1 \cdots \boldsymbol{r}_{K_P}] \in \mathbb{C}^{N\times K_P}$ is the primary data matrix and $\boldsymbol{R}_S = [\boldsymbol{r}_{K_P+1} \cdots \boldsymbol{r}_{K_P+K_S}] \in \mathbb{C}^{N\times K_S}$ is the secondary data matrix. Moreover let $\boldsymbol{Q} = [\boldsymbol{q}_1 \cdots \boldsymbol{q}_{K_P}] \in \mathbb{C}^{q\times K_P}$, $\boldsymbol{\alpha} = [\alpha_1 \cdots \alpha_{K_P}] \in \mathbb{C}^{1\times K_P}$, and $K = K_P + K_S$.

The above assumptions imply that the probability density function (pdf) of $\boldsymbol{R}$ can be written as

$$f_0(\boldsymbol{R};\boldsymbol{M},\boldsymbol{Q}) = \left(\frac{1}{\pi^N \det[\boldsymbol{M}]}\right)^K e^{-\mathrm{tr}\left[\boldsymbol{M}^{-1}\left(\boldsymbol{T}_0+\boldsymbol{S}\right)\right]} \tag{2}$$

under H_0 and

$$f_1(\boldsymbol{R};\boldsymbol{M},\boldsymbol{Q},\boldsymbol{p},\boldsymbol{\alpha}) = \left(\frac{1}{\pi^N \det[\boldsymbol{M}]}\right)^K e^{-\mathrm{tr}\left[\boldsymbol{M}^{-1}\left(\boldsymbol{T}_1+\boldsymbol{S}\right)\right]} \tag{3}$$

under H_1, where $\det(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the determinant and the trace of a square matrix, respectively, $\boldsymbol{S} = \boldsymbol{R}_S\boldsymbol{R}_S^\dagger \in \mathbb{C}^{N\times N}$ is K_S times the sample covariance matrix based on secondary data, with † being conjugate transpose, and $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ are given by

$$\boldsymbol{T}_0 = \sum_{k\in\Omega_P} (\boldsymbol{r}_k - \boldsymbol{J}\boldsymbol{q}_k)(\boldsymbol{r}_k - \boldsymbol{J}\boldsymbol{q}_k)^\dagger$$

$$\boldsymbol{T}_1 = \sum_{k\in\Omega_P} (\boldsymbol{r}_k - \alpha_k\boldsymbol{H}\boldsymbol{p} - \boldsymbol{J}\boldsymbol{q}_k)(\boldsymbol{r}_k - \alpha_k\boldsymbol{H}\boldsymbol{p} - \boldsymbol{J}\boldsymbol{q}_k)^\dagger.$$

3.1. One-Step GLRT-based Detector

We now derive the GLRT based upon primary and secondary data which is tantamount to the following decision rule

$$\Lambda(\boldsymbol{R}) = \frac{\max\limits_{\boldsymbol{\alpha}}\max\limits_{\boldsymbol{p}}\max\limits_{\boldsymbol{Q}}\max\limits_{\boldsymbol{M}} f_1(\boldsymbol{R};\boldsymbol{M},\boldsymbol{Q},\boldsymbol{p},\boldsymbol{\alpha})}{\max\limits_{\boldsymbol{Q}}\max\limits_{\boldsymbol{M}} f_0(\boldsymbol{R};\boldsymbol{M},\boldsymbol{Q})} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{4}$$

where γ is the threshold value to be set in order to ensure the desired Probability of False Alarm (P_{fa}).

It can be shown that optimization under the H_0 hypothesis leads to the following compressed likelihood function

$$f_0(\boldsymbol{R};\widehat{\boldsymbol{M}},\widehat{\boldsymbol{Q}}) = \left(\frac{K}{e\pi}\right)^{NK} \frac{1}{\det^K[\boldsymbol{S}]\det^K[\boldsymbol{A}]}, \tag{5}$$

where $\boldsymbol{A} = \boldsymbol{I}_{K_P} + (\boldsymbol{S}^{-1/2}\boldsymbol{R}_P)^\dagger(\boldsymbol{I}_N - \boldsymbol{P}_{\boldsymbol{J}_S})(\boldsymbol{S}^{-1/2}\boldsymbol{R}_P)$, with, in turn, $\boldsymbol{I}_N$ denoting the N-dimensional identity matrix, $\boldsymbol{J}_S = \boldsymbol{S}^{-1/2}\boldsymbol{J}$, and $\boldsymbol{P}_{\boldsymbol{K}}$ the projection matrix onto the subspace spanned by the columns of $\boldsymbol{K}$.

On the other hand, optimization under the H_1 hypothesis gives the following compressed likelihood function

$$f_1(\boldsymbol{R};\widehat{\boldsymbol{M}},\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{p}},\widehat{\boldsymbol{\alpha}}) = \left(\frac{K}{e\pi}\right)^{NK} \frac{(\det[\boldsymbol{S}]\det[\boldsymbol{A}])^{-K}}{(1-\lambda_{\max}\{\boldsymbol{B}\})^K}, \tag{6}$$

where $\lambda_{\max}\{\cdot\}$ denotes the maximum eigenvalue of the matrix argument and

$$\boldsymbol{B} = \boldsymbol{P}_{\boldsymbol{H}'_S}\boldsymbol{Z}^\dagger\boldsymbol{S}^{-1/2}\boldsymbol{R}_P\boldsymbol{A}^{-1}\boldsymbol{R}_P^\dagger\boldsymbol{S}^{-1/2}\boldsymbol{Z}\boldsymbol{P}_{\boldsymbol{H}'_S},$$

with

- $\boldsymbol{Z} \in \mathbb{C}^{N\times(N-q)}$ a slice of a unitary matrix satisfying $\boldsymbol{I}_N - \boldsymbol{P}_{\boldsymbol{J}_S} = \boldsymbol{Z}\boldsymbol{Z}^\dagger$;
- $\boldsymbol{H}'_S = \boldsymbol{Z}^\dagger\boldsymbol{S}^{-1/2}\boldsymbol{H}$.

Substituting the above results (5) and (6) into test (4) yields

$$\Lambda(\boldsymbol{R}) = \lambda_{\max}\{\boldsymbol{B}\} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{7}$$

3.2. Two-Step GLRT-based Detector

This subsection is devoted to the derivation of an ad hoc detector for problem (1) based upon the two-step GLRT-based design procedure. Specifically, first we derive the GLRT detector assuming that $\boldsymbol{M}$ is known. Then we come up with a fully-adaptive detector by replacing $\boldsymbol{M}$ with $\boldsymbol{S}$. To this end, note that the pdf of $\boldsymbol{R}_P$ is

$$f_0(\boldsymbol{R}_P;\boldsymbol{Q}) = \left(\frac{1}{\pi^N \det[\boldsymbol{M}]}\right)^{K_P} e^{-\mathrm{tr}\left[\boldsymbol{M}^{-1}\boldsymbol{T}_0\right]},$$

under H_0 and

$$f_1(\boldsymbol{R}_P;\boldsymbol{Q},\boldsymbol{p},\boldsymbol{\alpha}) = \left(\frac{1}{\pi^N \det[\boldsymbol{M}]}\right)^{K_P} e^{-\mathrm{tr}\left[\boldsymbol{M}^{-1}\boldsymbol{T}_1\right]},$$

under H_1. The GLRT for known $\boldsymbol{M}$ is given by

$$\Lambda(\boldsymbol{R}_P) = \frac{\max\limits_{\boldsymbol{\alpha}}\max\limits_{\boldsymbol{p}}\max\limits_{\boldsymbol{Q}} f_1(\boldsymbol{R}_P;\boldsymbol{Q},\boldsymbol{p},\boldsymbol{\alpha})}{\max\limits_{\boldsymbol{Q}} f_0(\boldsymbol{R}_P;\boldsymbol{Q})} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{8}$$

It is not difficult to show that test (8) can be re-cast as

$$e^{\lambda_{\max}\left\{\boldsymbol{P}_{\boldsymbol{H}'_w}\boldsymbol{Z}_w^\dagger\boldsymbol{M}^{-1/2}\boldsymbol{R}_P\boldsymbol{R}_P^\dagger\boldsymbol{M}^{-1/2}\boldsymbol{Z}_w\boldsymbol{P}_{\boldsymbol{H}'_w}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma,$$

where

- $\boldsymbol{Z}_w \in \mathbb{C}^{N\times(N-q)}$ is a slice of a unitary matrix satisfying $\boldsymbol{I}_N - \boldsymbol{P}_{\boldsymbol{J}_w} = \boldsymbol{Z}_w\boldsymbol{Z}_w^\dagger$, with $\boldsymbol{J}_w = \boldsymbol{M}^{-1/2}\boldsymbol{J}$;
- $\boldsymbol{H}'_w = \boldsymbol{Z}_w^\dagger\boldsymbol{M}^{-1/2}\boldsymbol{H}$.

The logarithmic GLRT is

$$\log\Lambda(\boldsymbol{R}_P) = \lambda_{\max}\left\{\boldsymbol{C}(\boldsymbol{M})\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{9}$$

where

$$\boldsymbol{C}(\boldsymbol{M}) = \boldsymbol{P}_{\boldsymbol{H}'_w}\boldsymbol{Z}_w^\dagger\boldsymbol{M}^{-1/2}\boldsymbol{R}_P\boldsymbol{R}_P^\dagger\boldsymbol{M}^{-1/2}\boldsymbol{Z}_w\boldsymbol{P}_{\boldsymbol{H}'_w}.$$

Note that $\boldsymbol{C}(\boldsymbol{M})$ depends on $\boldsymbol{M}$ also through $\boldsymbol{Z}_w$ and $\boldsymbol{P}_{\boldsymbol{H}'_w}$. Finallly, plugging $\boldsymbol{S}$ in place of $\boldsymbol{M}$ into (9) yields the fully-adaptive rule

$$\lambda_{\max}\left\{\boldsymbol{C}(\boldsymbol{S})\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{10}$$

where

$$\boldsymbol{C}(\boldsymbol{S}) = \boldsymbol{P}_{\boldsymbol{H}'_S}\boldsymbol{Z}^\dagger\boldsymbol{S}^{-1/2}\boldsymbol{R}_P\boldsymbol{R}_P^\dagger\boldsymbol{S}^{-1/2}\boldsymbol{Z}\boldsymbol{P}_{\boldsymbol{H}'_S}.$$

As a final remark, it can be shown that in the case $r + q = N$, receivers (7) and (10) become equivalent; however, the proof of such a statement is not reported here for the lack of space.

4. PERFORMANCE ASSESSMENT

Since closed-form expressions for the Probability of Detection (P_d) and the P_{fa} are not available, we resort to standard Monte Carlo counting techniques. More precisely, in order to evaluate the threshold necessary to ensure a preassigned value of P_{fa} and P_d, we resort to $100/P_{fa}$ and 10^4 independent trials, respectively.

We randomly generate the columns of $\boldsymbol{H}$ and $\boldsymbol{J}$ at each iteration of the Monte Carlo simulation as independent and identically distributed (iid) complex normal vectors with zero mean and identity covariance matrix. In addition, the interference coordinates $\boldsymbol{q}_k$, $k \in \Omega_P$, are iid complex normal vectors with zero mean and covariance matrix given by $\sigma_J^2 \boldsymbol{I}_q$, where σ_J^2 is the power of the interference and $\boldsymbol{I}_q$ denotes the q-dimensional identity matrix. The vector $\boldsymbol{p}$ is a complex normal vector with zero mean and covariance matrix given by $\boldsymbol{I}_r$. In addition, above vectors are each other independent. Finally, $|\alpha_k| = |\alpha|$, $k \in \Omega_P$.

As to the noise, it is modeled as an exponentially correlated complex normal vector with one-lag correlation coefficient ρ, namely the (i, j)-th element of the covariance matrix $\boldsymbol{M}$ is given by $\sigma_n^2 \rho^{|i-j|}$, $i, j = 1, \ldots, N$, with $\rho = 0.95$ and $\sigma_n^2 = 1$.

The P_{fa} is set to 10^{-4} and the Signal-to-Noise Ratio (SNR) is defined as

$$\text{SNR} = \frac{\sum_{k=1}^{K_P} |\alpha_k|^2 E\left[\|\boldsymbol{H}\boldsymbol{p}\|^2\right]}{N\sigma_n^2}, \tag{11}$$

where $|\cdot|$ denotes the modulus of a complex number, $E[\cdot]$ denotes statistical expectation, and $\|\cdot\|$ is the Euclidean norm of an N-dimensional vector over the complex field. Note that

$$E\left[\|\boldsymbol{H}\boldsymbol{p}\|^2\right] = \text{tr}\left\{E\left[\boldsymbol{H}^\dagger \boldsymbol{H}\boldsymbol{p}\boldsymbol{p}^\dagger\right]\right\} = Nr.$$

Finally, we assume that the Interference-to-Noise Power Ratio (INR), defined as σ_J^2/σ_n^2, is equal to 30 dB.

Figures 1-4 compare the performance, in terms of P_d versus SNR, of the plain GLRT (7) and the ad hoc detector (10).

In particular, Figures 1 and 2 assume $N = 8$, $K_P = 8$, and $K_S = 16$, and different values of parameters r and q. More precisely, in Figure 1 we set $r = 2$ and $q = 2$ while in Figure 2 we set $r = 4$ and $q = 2$. In both figures, detector (7) performs slightly better than detector (10), but note that when r increases the horizontal displacement between the corresponding curves reduces (the gain reduces from about 1.1 dB to 0.4 dB). Figures 3 and 4 refer to $N = 16$, $K_P = 16$, and $K_S = 16$, and different values of parameters r and q; more precisely, Figure 3 assumes $r = 2$ and $q = 2$ while Figure 4 refers to $r = 4$ and $q = 2$. It is apparent that in the scenarios of Figures 3 and 4 the GLRT performs much better than the ad hoc detector.

Summarizing, figures highlight that detector (7) performs better than detector (10) when $r + q < N$, with a gain depending on simulation parameters.

5. CONCLUSIONS

In this paper we have implemented two GLRT-based direction detectors capable to operate in homogeneous Gaussian noise and subspace interference. To this end, we have supposed that a set of noise only data is available and that the useful target and the interference belong to known subspaces of the observables. The preliminary performance assessment highlights that the plain GLRT performs better than the ad hoc detector, although at the price of a certain increase of the computational complexity. However, simulation studies also seem to indicate that the gains are in the order of 1 dB (or less) when $K_S \geq 2N$. A definite validation of previous results will require a thorough performance assessment also in comparison with other "robust" techniques capable to take into account possible uncertainties on the actual steering vector of the target.

6. REFERENCES

[1] E. Conte, A. De Maio, G. Ricci, "GLRT-based Adaptive Detection Algorithms for Range-Spread Targets," *IEEE Trans. on Signal Processing*, Vol. 49, No. 7, pp. 1336-1348, July 2001.

[2] E. Conte, A. De Maio, G. Ricci, "CFAR Detection of Distributed Targets in Non-Gaussian Disturbance," *IEEE Trans. on Aerospace and Electronics Systems*, Vol. 38, No. 2, pp. 612-621, April 2002.

[3] E. Conte, A. De Maio, G. Ricci, "Recursive Estimation of the Covariance Matrix of a Compound-Gaussian Process and Its Application to Adaptive CFAR Detection," *IEEE Trans. on Signal Processing*, Vol. 50, No. 8, pp. 1908-1915, August 2002.

[4] S. Kraut, L. L. Scharf, L. T. McWhorter, "Adaptive Subspace Detectors," *IEEE Trans. on Signal Processing*, Vol. 49, No. 1, pp. 1-16, January 2001.

[5] E. J. Kelly, K. Forsythe, "Adaptive Detection and Parameter Estimation for Multidimensional Signal Models," Lincoln Lab, MIT, Lexington, Tech. Rep. No. 848, April 19, 1989.

[6] O. Besson, L. L. Scharf, F. Vincent, "Matched Direction Detectors," in *Proc of ICASSP 2005*, Vol. IV, pp. 977-980.

[7] O. Besson, L. L. Scharf, "CFAR Matched Direction Detector," submitted to *IEEE Trans. on Signal Processing*.

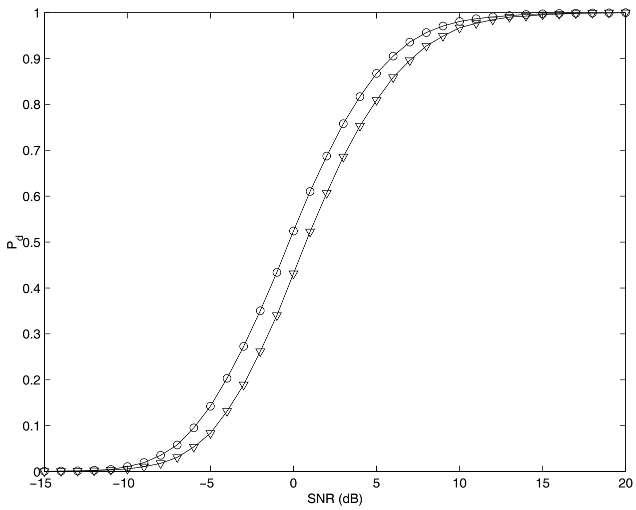

Fig. 1. P_d vs SNR with $N = 8$, $K_P = 8$, $K_S = 16$, $r = 2$, and $q = 2$ for detector (7) (circle marker) and detector (10) (triangle marker).

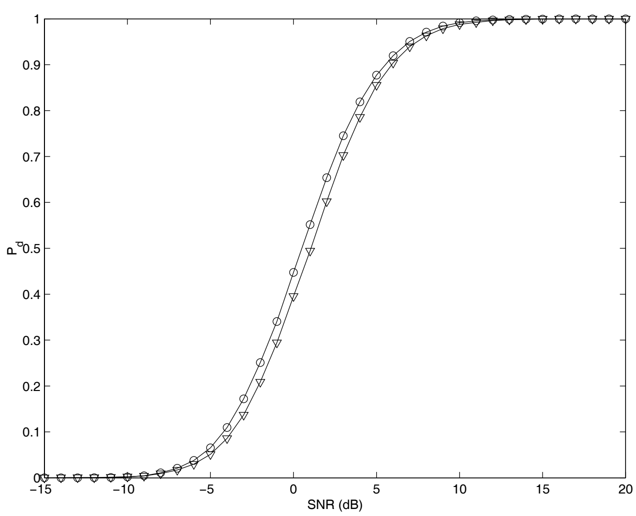

Fig. 2. P_d vs SNR with $N = 8$, $K_P = 8$, $K_S = 16$, $r = 4$, and $q = 2$ for detector (7) (circle marker) and detector (10) (triangle marker).

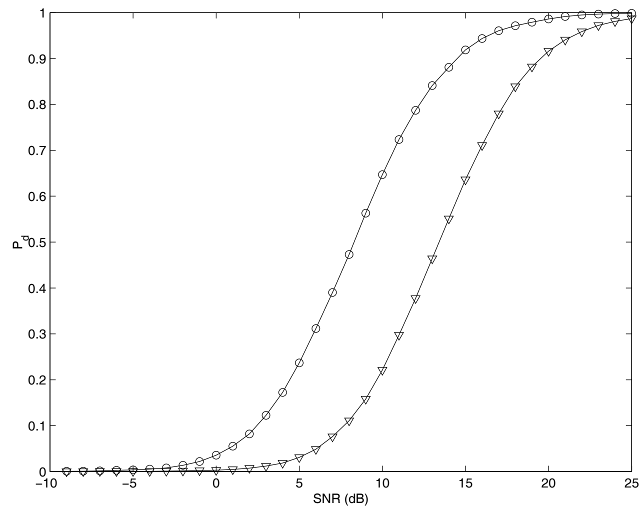

Fig. 3. P_d vs SNR with $N = 16$, $K_P = 16$, $K_S = 16$, $r = 2$, and $q = 2$ for detector (7) (circle marker) and detector (10) (triangle marker).

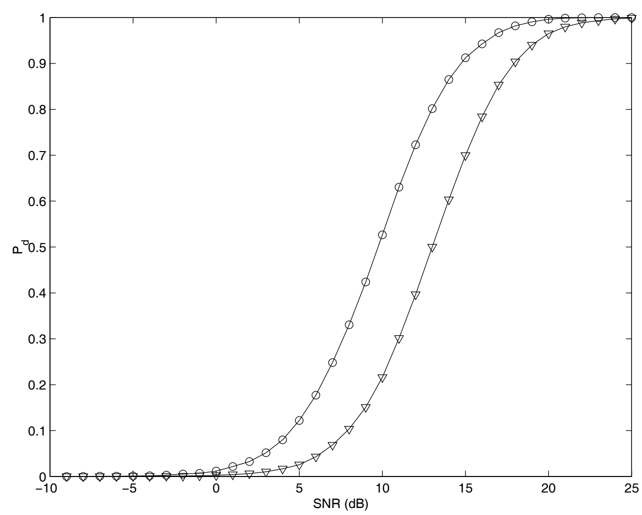

Fig. 4. P_d vs SNR with $N = 16$, $K_P = 16$, $K_S = 16$, $r = 4$, and $q = 2$ for detector (7) (circle marker) and detector (10) (triangle marker).