SYMMETRIC ORTHOGONAL COMPLEX-VALUED FILTER BANK DESIGN BY SEMIDEFINITE PROGRAMMING

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ABSTRACT

A new design method for complex-valued two-channel FIR filter banks with both orthogonality and symmetry properties is developed. Based on a novel linear matrix inequality (LMI) characterization of trigonometric curves, the optimal design of the perfect reconstruction filter bank is reformulated as a semi-definite programme. The dimension of the resulting semi-definite programme is further reduced by exploiting the strong convex duality. Consequently, the globally optimal solution can be effectively found for any practical filter length and desired regularity order.

1. INTRODUCTION

Orthogonal filter banks with symmetric FIR filters are of great interest in certain applications of image and video processing. The symmetry property of filters is important for effectively handling boundary distortions of finite length signals [10]. On the other hand, orthogonal filter banks preserve the energy of the input signal in the subbands, which guarantees that errors arising from quantization or transmission will not be amplified. Moreover, the orthogonality property usually leads to high energy compaction [9]. However, the real-valued two-channel filter banks with simultaneous orthogonality and symmetry do not exist except the trivial Haar filters with two coefficients [7], [9]. In contrast, the nontrivial orthogonal and symmetric complex-valued filter banks do exist and they are capable of providing even more potentially beneficial properties. They produce orthogonal and symmetric complex wavelets, which can offer both shift invariance and good directional selectivity, compared to shift variance and poor directional selectivity of real-valued wavelets [5]. Furthermore, they can be applied to complex systems such as radar signal, discrete multi-tone modulation (DMT) signal [3].

A known method for designing complex-valued filter banks is based on the lattice structure. It requires solutions of highly nonlinear complex equations that are not practically solvable for large length filters [3]. This paper proposes a new design method for complex-valued two-channel FIR filter banks where both orthogonality and symmetry properties are simultaneously prevailed. The conditions for perfect reconstruction, symmetry and regularity are completely characterized by linear matrix inequality constraints of (convex) semi-definite programming (SDP). In other words, our design problem is effectively reformulated as a SDP. Furthermore, the convex duality allows us to reduce it to yet another SDP but with much smaller dimension. Subsequently, the globally optimal solution can be efficiently computed for, effectively, any filter length and desired regularity order.

The organization of this paper is as follows. After the Introduction, Section 2 gives the optimization formulation for the filter bank design. Its conversion to SDP for attractive computation is described in Section 3, whose viability is confirmed by numerical examples in Section 4. The conclusions are given in Section 5.

The notations of the paper are rather standard. In particular, the notation $X \geq 0$ denotes a (symmetric) positive semi-definite matrix. It is a trivial fact that the dimension of the space of all $N \times N$-symmetric matrices is $N(N+1)/2$; the inner product $\langle X, Y\rangle$ of the matrices X and Y is given by Trace(XY), so $\langle X, Y\rangle \geq 0$ for $X \geq 0,\ Y \geq 0$. For a given set $C \subset \mathbb{R}^N$, its convex hull (conic hull), denoted by conv(C) (cone(C)), is the smallest convex set (cone) in $\mathbb{R}^N$ that contains C. The polar set of C is the cone $C^* = \{x :\ \langle x, y\rangle \geq 0\, \forall\, y \in C\} \subset \mathbb{R}^N$. It is straightforward to see that $C^* = (\text{conv}(C))^* = (\text{cone}(C))^*$ and if C is a closed convex cone then $C = (C^*)^*$. Further, the tilde accent on a function $H(z)$ is defined as $\tilde{H}(z) = H_*(z^{-1})$, where asterisk subscript $(_*)$ denotes the conjugation of coefficients without conjugating z. e_1 stands for the unit vector $e_1 = (1\ 0\ 0\ ...\ 0)^T$. With some abuse of notation, we use $G(\omega)$ to refer to $G(e^{j\omega})$ for short.

2. MATHEMATICAL MODEL OF ORTHOGONAL AND SYMMETRIC FILTER BANKS

A two-channel maximally decimated uniform filter bank is illustrated by Fig. 1. An analysis filter bank with the lowpass filter $H_0(z)$ and highpass filter $H_1(z)$ decomposes the input signal $X(z)$ into the subband signals $X_0(z)$ and $X_1(z)$. This is followed by a synthesis filter bank with the lowpass filter $F_0(z)$ and highpass filter $F_1(z)$, which reconstructs the output signal $\hat{X}(z)$ from the subband signals. It is easily shown that the output $\hat{X}(z)$ of the two-channel filter bank is given by

$$\hat{X}(z) = \frac{1}{2}\left[F_0(z)H_0(z) + F_1(z)H_1(z)\right]X(z) + \frac{1}{2}\left[F_0(z)H_0(-z) + F_1(z)H_1(-z)\right]X(-z).$$

If $\hat{X}(z) = cz^{-\ell}X(z)$, the filter bank has perfect-reconstruction property. The perfect reconstruction filter bank is designed by constructing the analysis and synthesis filters from the lowpass proto-

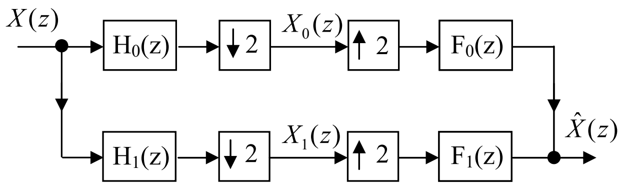


Fig. 1. Maximally decimated two channel filter bank.

type filter $H(z)$ according to

$$\begin{aligned} H_0(z) = H(z), \quad H_1(z) = -z^{-N}\tilde{H}(-z), \\ F_0(z) = z^{-N}\tilde{H}_0(z), \quad F_1(z) = z^{-N}\tilde{H}_1(z). \end{aligned} \tag{1}$$

The above choices of filters lead to the perfect filter bank with orthogonality if and only if the prototype filter satisfies the following condition

$$H(z)\tilde{H}(z) + H(-z)\tilde{H}(-z) = 1 \tag{2}$$

for some odd N ($N = 2L + 1$).

Next, from the relations (1), the symmetry of all filters is clearly guaranteed by that imposed on the prototype filter $H(z)$

$$h_k = h_{N-k},\ k = 0, 1, ..., N \tag{3}$$

for the complex-valued coefficients $h = (h_0, h_1, ..., h_N)^T$ of $H(z)$.

In short, the problem of designing an orthogonal and symmetric filter bank is down to designing a prototype filter $H(z)$ satisfying the orthogonality condition (2) and the symmetry condition (3), where the former is highly nonlinear and the later is linear constraints in the filter coefficients h.

Introduce the product filter

$$G(z) = H(z)\tilde{H}(z) = \sum_{k=-N}^{N} \bar{g}_k z^{-k} \tag{4}$$

which is a positive real filter with symmetric coefficients $\bar{g}_k = \bar{g}_{-k}$. For simplicity of presentation, we define

$$g = (g_0, g_1, ..., g_N)^T = (\bar{g}_0, 2\bar{g}_1, ..., 2\bar{g}_N)^T \in \mathbb{R}^{N+1}.$$

Then, in term of the product filter $G(z)$, the condition (2) is just the following linear constraints in the coefficients g

$$g_{2k} = \frac{1}{2}\delta(k),\ k = 0, 1, ..., L. \tag{5}$$

It can be compactly written as

$$Ag = b \tag{6}$$

where

$$\begin{gathered} A(i,j) = \delta(2i - j),\ i = 0, 1, ..., L,\ j = 0, 1, ..., N, \\ b = (1, 0, ..., 0)^T \in \mathbb{R}^{L+1}. \end{gathered}$$

However, not any positive real filter $G(z)$ can be factorized in form (4) with $H(z)$ satisfying the symmetry condition (3). In fact, in view of (3) and (4), $G(z)$ must have the following specific form

$$G(\omega) = \sum_{k=0}^{N} g_k \cos(k\omega) = \left| \sum_{k=0}^{L} h_k \left[e^{-jk\omega} + e^{-j(N-k)\omega} \right] \right|^2, \tag{7}$$

for some $(h_0, ..., h_L) \in \mathbb{C}^{L+1}$.

In addition, in certain applications, the regularity of lowpass filters is required. The filter $H(z)$ is said to be p-regular if it has p multiple zeros at $z = -1$, or equivalently $G(\omega)$ has $2p$ multiple zeros at $\omega = \pi$, which are linear constraints in the filter coefficients g:

$$\sum_{k=0}^{N} k^i (-1)^k g_k = 0 \qquad i = 0, 2, 4, ..., 2p - 2, \tag{8}$$

or equivalently,

$$c^{(i)T} g = 0, \qquad i = 0, 1, ..., p - 1 \tag{9}$$

with $c^{(i)} \in \mathbb{R}^{N+1}$, $c^{(i)}(k) = k^{2i}(-1)^k$, $k = 0, 1, ..., N$.

Finally, the following standard constrained specifications on the product filter $G(z)$ are imposed to get the smooth frequency selectivities on given stopband $[0, \omega_p]$ and passband $[\omega_s, \pi]$ with $\omega_p = (1 - \epsilon)\frac{\pi}{2}$, $\omega_s = (1 + \epsilon)\frac{\pi}{2}$, where ϵ is a positive constant depending on the required transition width:

- The objective function is to minimize the square error

$$\begin{aligned} E(g) &= \int_0^{\omega_p} |G(\omega) - 1|^2 d\omega + \int_{\omega_s}^{\pi} |G(\omega)|^2 d\omega \\ &= g^T Q g + q^T g + r \end{aligned} \tag{10}$$

where $Q = \int_0^{\omega_p} \mathcal{T}_N(\omega) d\omega + \int_{\omega_s}^{\pi} \mathcal{T}_N(\omega) d\omega,$

$$q = -2\int_0^{\omega_p} \varphi_N(\omega) d\omega, \quad r = \omega_p.$$

- The peak-error constraints

$$\begin{aligned} 1 - \delta \le G(\omega) \le 1, \quad &\forall\, \omega \in [0, \omega_p] \\ 0 \le G(\omega) \le \delta, \quad &\forall\, \omega \in [\omega_s, \pi] \end{aligned} \tag{11}$$

are fully satisfied.

In summary, designing the two-channel filter bank with orthogonality, symmetry and regularity properties is equivalent to minimizing the convex quadratic objective function (10) subject to the linear constraints (6), (9), the nonlinear constraint (7), and the semi-infinite linear constraints (11) in the variable g:

$$\min_g E(g) \quad \text{s.t.} \quad (6), (7), (9), (11). \tag{12}$$

The next section is devoted to reducing the nonlinear constraint (7) and semi-infinite constraints into LMIs so our filter bank design is in fact formulated as a SDP. Furthermore, the convex duality is employed to reduce the dimension of this SDP for practically efficient computation.

3. CONVERSION TO SEMI-DEFINITE PROGRAMMING

The constraint (7) is highly non-linear relationship between the product filter coefficients g and the symmetric complex prototype filter coefficients h. It can be seen that if h in (7) is restricted real only then the set of all $g \in R^{N+1}$ satisfying (7) is nonconvex and thus there is no way to express (7) as a LMI constraint. Interestingly enough, the situation is gradually changed when $h \in \mathbb{C}^{L+1}$: the set of all such g will be convex and described by LMI constraints as we show right now.

With the introduced function

$$\varphi_N(\omega) = (1, \cos(\omega), \cos(2\omega), ..., \cos(N\omega))^T, \omega \in [0, \pi],$$

the $N-$th order trigonometric moment matrix $\mathcal{T}_L(\omega)$ is defined as [8]

$$\mathcal{T}_N(\omega) = \varphi_N(\omega)\varphi_N^T(\omega). \tag{13}$$

The matrix $T_N(y)$ depending on $y = (y_0, y_1..., y_{2N})^T$ results from $\mathcal{T}_N(\omega)$ through the variable change

$$\cos k\omega \to y_k, \; k = 0, 1, ..., 2N, \tag{14}$$

so

$$T_N(y) = \begin{bmatrix} y_0 & y_1 & \dots & y_N \\ y_1 & \frac{y_2+y_0}{2} & \dots & \frac{y_{N+1}+y_{N-1}}{2} \\ \dots & \dots & \dots & \dots \\ y_N & \frac{y_{N+1}+y_{N-1}}{2} & \dots & \frac{y_{2N}+y_0}{2} \end{bmatrix}$$

and $\mathcal{T}_N(\omega) = T_N(\varphi_N(\omega))$.
Furthermore, we also define $\mathcal{T}_{\ell N}(\omega) = \cos \ell\omega \mathcal{T}_N(\omega)$ and accordingly, $T_{\ell N}(y)$ is derived from $\mathcal{T}_{\ell N}(\omega)$ by variable change (14) which results in $\mathcal{T}_{\ell N}(\omega) = T_{\ell N}(\phi_N(\omega))$.

The following theorem shows that the nonlinear constraint (7) is indeed recast as a LMI.

Theorem 1 *The set*

$$C = \Big\{ g \in \mathbb{R}^{N+1} : \; \exists \, h = h_R + \jmath h_I \in \mathbb{C}^{L+1} s.t. \; (7) \Big\} \tag{15}$$

is convex as it can be expressed by the LMI constraint

$$C = \Big\{ g \in \mathbb{R}^{N+1} : \; G(\omega) \equiv \langle X, \mathcal{T}_L(\omega) + \mathcal{T}_{1L}(\omega) \rangle \quad \textit{for some } X \geq 0 \Big\}. \tag{16}$$

The polar cone C^* is also expressed by a LMI constraint

$$C^* = \Big\{ y \in \mathbb{R}^{N+1} : \; T_L(y) + T_{1L}(y) \geq 0 \Big\}. \tag{17}$$

It should be clarified that by comparing g_i with the coefficient of the same "power" $\cos(i\omega)$ in $\langle X, \mathcal{T}_L(\omega) + \mathcal{T}_{1L}(\omega) \rangle$, linear relations between g and X in (16) are easily established and they together with the constraint $X \geq 0$ constitute the LMI constraint for describing C defined by (16).

Next, to handle the semi-infinite trigonometric constraints (11) we simply use the result of [8]. A trigonometric curve $C_{a,b}$ is defined as

$$C_{a,b} = \Big\{ \varphi_N(\omega) : \cos\omega \in \big[\cos a, \cos b\big] \Big\} \subset \mathbb{R}^{N+1}, \tag{18}$$

with its polar

$$C^*_{a,b} = \Big\{ u \in \mathbb{R}^{N+1} : \langle u, v \rangle \; \geq 0 \quad \forall v \in C_{a,b} \Big\}. \tag{19}$$

Theorem 2 *[8] Define the following LMI constraint in the variables* $y = (y_0, y_1, ..., y_N) \in \mathbb{R}^{N+1}$

$$\cos bT_L(y) \geq T_{1L}(y) \geq \cos aT_L(y) \quad \textit{for } N = 2L+1. \tag{20}$$

Then, the convex hull of the trigonometric curve $C_{a,b}$ defined by (18) is fully characterized by LMI constraints

$$conv(C_{a,b}) = \{(y_0, y_1, y_2, ..., y_N) : (20), y_0 = 1\}. \tag{21}$$

Consequently, the conic hull of $convC_{a,b}$ is defined by

$$cone(C_{a,b}) = \{(y_0, y_1, y_2, ..., y_N) : \; (20)\}).$$

Thus, the semi-infinite trigonometric constraints (11) are described by the following LMI constraints

$$\beta_i g + d_i \in C_i^*, \qquad i = 1, 2, 3, 4 \tag{22}$$

where

$$C_1^* = C_2^* = C^*_{\omega_p, 0}, \qquad C_3^* = C_4^* = C^*_{\pi, \omega_s},$$
$$\beta_1 = \beta_3 = 1, \qquad \beta_2 = \beta_4 = -1,$$
$$d_1 = (-1+\delta)e_1, d_2 = e_1, d_3 = 0, d_4 = \delta e_1.$$

Summing up, the optimization problem (12) is reformulated as the following

$$\min_g g^T Q g + q^T g \qquad \text{s.t. } (6), (9), (16), (22) \tag{23}$$

which is a SDP because (6), (9) are linear constraints, and (16), (22) are LMIs. It should be noted that (22) is four LMI constraints involving 8 positive semi-definite matrix additional variables of dimension $(L+1) \times (L+1)$ (see [8] for more details on equivalent LMI constraints for the set $C^*_{a,b}$ defined by (19)) and (16) is also a LMI involving one symmetric positive semi-definite matrix variable X of dimension $(L+1) \times (L+1)$. Thus the total number of scalar variables for SDP (23) is $2L + 1 + 9(L+1)(L+2)/2$, which may be too high for large L.

Like [8], we can reduce the variable dimension for the SDP (23) through the convex duality. Using the Lagrange multiplier method, the optimization problem (23) can be rewritten as the following SDP

$$\max_{\lambda, \alpha_i, y^{(i)}, \eta} \quad -b^T\lambda - \sum_{i=0}^{4} d_i^T y^{(i)} - \eta \tag{24}$$

subject to

$$\begin{bmatrix} \eta & * \\ q + A^T\lambda + \sum_{i=0}^{p-1} \alpha_i c^{(i)} - \sum_{i=0}^{4} \beta_i y^{(i)} & 4Q \end{bmatrix} \geq 0,$$
$$T_L(y^{(0)}) + T_{1L}(y^{(0)}) \geq 0,$$
$$T_L(y^{(i)}) \geq T_{1L}(y^{(i)}) \geq cos(\omega_p) T_L(y^{(i)}) \quad i = 1, 2,$$
$$cos(\omega_s) T_L(y^{(i)}) \geq T_{1L}(y^{(i)}) \geq -T_L(y^{(i)}) \quad i = 3, 4,$$

where $\beta_0 = 1$ and $d_0 = 0$.
Note that in contrast to the primal optimization problem (23), the SDP (24) involves 5 variables $y^{(i)}$ of dimension $2L+2$, one variable λ of dimension $L+1$ and p other scalar variables (α_i and η) so the dimension of its variable is not an issue in computational implementation even for large L of the thousand magnitude.

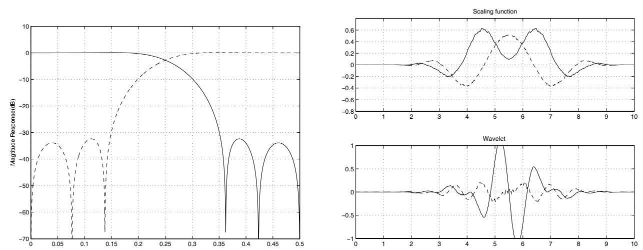
Once the optimal solutions $y_*^{(i)}$, α_{i*} and λ_* of the dual problem (24) have been found, the optimal solution g_* of the primal (23) is retrieved by the following equation:

$$g_* = -\frac{1}{2} Q^{-1} \left(q + A^T\lambda_* + \sum_{i=0}^{p-1} \alpha_{i*} c^{(i)} - \sum_{i=0}^{4} \beta_i y_*^{(i)} \right). \tag{25}$$

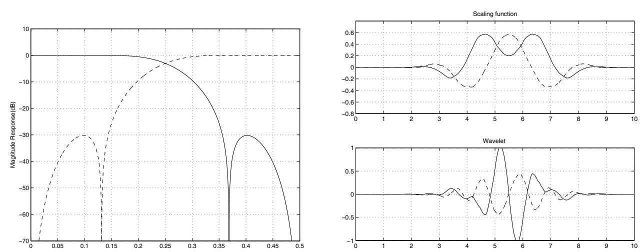
Finally, the optimal prototype filter $H(z)$ satisfying (7) is easily recovered from the optimal product filter $G(z)$ by the spectral factorization (see [2, 10]).

4. DESIGN EXAMPLES

Example 1: A symmetric orthogonal complex-valued two channel filter bank is designed using the above method. The lowpass prototype filter $H(z)$ has specifications: filter order $N = 11$, passband edge frequency $\omega_p = 0.3\pi$, stopband edge frequency $\omega_s = 0.7\pi$, and stopband attenuation $\delta = -25dB$. The lowpass filter of regularity order p=1, 3 is considered. The magnitude responses of the filter bank and accordingly generated scaling functions and wavelets are shown in Fig. 2. The wavelet and scaling functions are observed smoother as the regularity order increases.



(a) p=1 (b) p=1



(c) p=3 (d) p=3

Fig. 2. (a), (c) normalized frequency responses of the filter bank with N=11. (b), (d) the solid line shows the real part, and the dashed line shows the imaginary part of scaling functions and wavelets.

Example 2: In order to show the efficiency and flexibility of the above algorithm, another filter bank with longer filter length is designed. The lowpass prototype filter has $N = 63$, $\omega_p = 0.44\pi$, $\omega_s = 0.56\pi$, $\delta = -50dB$, and p=1,3. The outcome is again the magnitude responses of the filter bank and accordingly generated scaling functions and wavelets illustrated by Fig. 3.

5. CONCLUDING REMARKS

In this paper, a novel method for designing a symmetric orthogonal complex valued filter banks is presented. The key contribution is to show that the optimal design of this class of complex filter bank can be globally solved by a SDP of moderate size. Design examples show that the proposed method is really effective.

6. REFERENCES

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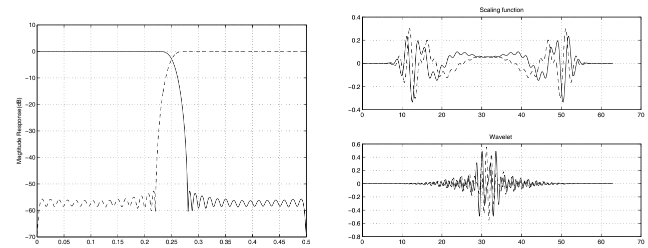
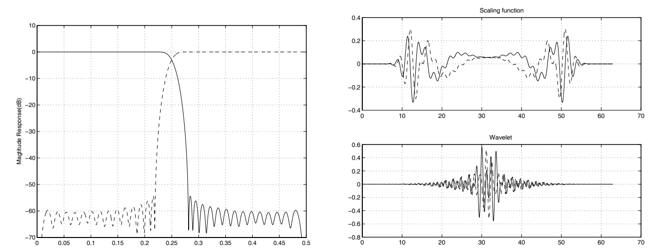


(a) p=1 (b) p=1



(c) p=3 (d) p=3

Fig. 3. (a), (c) normalized frequency responses of the filter bank with N=63. (b), (d) the solid line shows the real part, and the dashed line shows the imaginary part of scaling functions and wavelets.

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