Construction of Tight Filter Bank Frames*

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Abstract— In this paper, we present an explicit and numerically efficient formulae to construct a tight (paraunitary) FB frame from a given un-tight (non-paraunitary) FB frame. The derivation uses the well developed techniques from modern control theory, which results in the unified formulae for generic IIR and FIR FBs. These formulae involve only algebraic matrix manipulations and can be computed efficiently and reliably without the approximation required in the existing literature.

I. Introduction

A great deal of research has been devoted to the analysis and design of oversampled FBs with the advantages of increased design freedom, enhanced noise reduction, and improved capacity for signal and information representation [1]-[6]. In [1]-[3],[8], an elegant frame-theoretic approach is presented for the analysis and design of general oversampled FBs. Paraunitary FBs are more preferable in practice since the corresponding synthesis FBs are very easy to compute. Generally speaking, given an arbitrary analysis FB with polyphase representation $E(z)$, the canonical paraunitary FB can be obtained by $E(z)(\tilde{E}(z)E(z))^{-\frac{1}{2}}$ [1]. However, the above formula involves factorization and inversion of transfer matrices, for which there is a lack of efficient algorithms. Therefore, an approximation method is suggested in [1]. For paraunitary FIR FBs (or tight frames of finite dimensional spaces), various methods were proposed in [3], [9], [6].

In this paper, we will present explicit and numerically efficient formulae for constructing a tight (paraunitary) FB frame from an arbitrary un-tight (non-paraunitary) FB frame. The results provide directly computable formulation with the effort not exceeding algebraic matrix manipulations and not involving any approximation required in the existing literature, and hence resolve completely the problems discussed in the above. These new results are derived by the well-developed techniques in modern control theory [10].

II. Preliminaries

This sections reviews some preliminaries, in particular, the concepts of transfer matrices, state space realization, frames and oversampled filter banks. Please refer [1], [2], [10], [11] for more details.

The notation is standard. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The set of n-dimensional real (complex) vectors is denoted by $\mathbb{R}^n(\mathbb{C}^n)$. $\mathbb{R}^{m\times n}(\mathbb{C}^{m\times n})$ denotes the $m \times n$ real (complex) matrix set. For a matrix $A \in \mathbb{C}^{m\times n}$, A^* denotes its conjugate transpose.

A. Transfer matrices, state space descriptions and inner-outer factorization

Definition 1: For any rational transfer matrix $E(z) \in \mathbb{C}^{N\times M}$, if the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times M}$, $C \in \mathbb{R}^{N\times n}$ and $D \in \mathbb{R}^{N\times M}$ are such that $E(z) = D + C(zI - A)^{-1}B$, then (A, B, C, D) is called a state space realization of $E(z)$ and denoted as $E(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. The realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is minimal if the dimension of A is minimal.

For any transfer matrix $E(z)$, define $\tilde{E}(z) = E^*(z^{-1})$. A rational transfer matrix $N(z)$ is called inner if $N(z)$ is causal stable satisfying $\tilde{N}(z)N(z) = I$ for all $z = e^{j\theta}, \theta \in [0, 2\pi)$. Note that in such case $N^{\tilde{}}(z)$ is anti-causal and stable. Note also that the inner condition implies the paraunitary condition [1], [2].

The following lemma is a standard result from robust control theory, see eg Chapter 21 in [10].

Lemma 1: For $E(z) \in \mathbb{C}^{N\times M}$ with a minimal causal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, assume that $\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}$ has full column rank for all $\theta \in [0, 2\pi)$ and D has full column rank. Then $E(z)$ can be factorized in the following form

$$E(z) = N(z)M(z)^{-1} \tag{1}$$

where $M(z)$ and $N(z)$ are rational transfer matrices with $N(z)$ inner, and

$$\begin{bmatrix} M(z) \\ N(z) \end{bmatrix} := \left[\begin{array}{c|c} A + BF & BW^{-\frac{1}{2}} \\ \hline F & W^{-\frac{1}{2}} \\ C + DF & DW^{-\frac{1}{2}} \end{array}\right] \tag{2}$$

$$W = W^* = D^*D + B^*XB \tag{3}$$

$$F = -W^{-1}(B^*XA + D^*C) \tag{4}$$

and $X = X^* \geq 0$ is the unique stabilizing solution to

$$A^*XA - X + C^*C - (A^*XB + C^*D)W^{-1}(B^*XA + D^*C) = 0. \tag{5}$$

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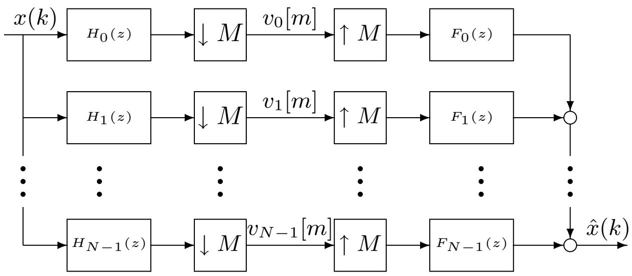

Fig. 1 Oversampled filter bank with $N \geq M$.

B. Filter banks, polyphase representation and frame theory

Consider an N-channel oversampled filter bank with decimation factor M as shown in Fig. 1. Assume that the analysis and synthesis filters, denoted respectively by $H_k(z)$ and $F_k(z)$, $k = 0, \ldots N-1$, are all linear and BIBO stable, and have the following form

$$H_k(z) = \sum_{n=-\infty}^{\infty} h_k[n]z^{-n} \text{ and } F_k(z) = \sum_{n=-\infty}^{\infty} f_k[n]z^{-n},$$

where $h_k[n]$ and $f_k[n]$ are impulse response coefficients of $H_k(z)$ and $F_k(z)$, respectively. Note that $H_k(z)$ is called FIR if there exists a finite number K such that $h_k[n] = 0$ for all $|n| \geq K$, and called IIR otherwise.

Denote $E(z)$ and $R(z)$ the polyphase matrix of the analysis filters $\{H_k(z)\}$ and the synthesis filters $\{F_k(z)\}$, respectively, where $E_{ij}(z) = \sum_{n=-\infty}^{\infty} h_i[nN-j]z^{-n}$, and $R_{ji}(z) = \sum_{n=-\infty}^{\infty} f_i[nN-j]z^{-n}$, for $i = 0, \ldots, N-1$ and $j = 0, \ldots, M-1$.

Definition 2: A sequence $\{\varphi_k[n]\}_{k=-\infty}^{\infty}$ in a Hilbert space $\ell^2(\mathbb{R})$ is a frame if there exist constants $\alpha, \beta > 0$ such that

$$\alpha||x||^2 \leq \sum_{k=-\infty}^{\infty} |<x, \varphi_k>|^2 \leq \beta||x||^2 \tag{6}$$

for any $x \in \ell^2(\mathbb{R})$, where $<x, \varphi_k>$ denotes the inner product of $x[n]$ and $\varphi_k[n]$. α and β are called the frame bounds of $\{\varphi_k[n]\}_{k=-\infty}^{\infty}$.

Definition 3: A frame $\{\varphi_k[n]\}_{k=-\infty}^{\infty}$ is tight if

$$\sum_{k=-\infty}^{\infty} |<x, \varphi_k>|^2 = \alpha||x||^2 \tag{7}$$

for any $x \in \ell^2(\mathbb{R})$.

If an FB satisfies PR with zero delay, then $\hat{x}[n] = x[n]$. For such FB, define

$$h_{k,m}[n] := h_k[mM-n] \text{ and } f_{k,m}[n] := f_k[n-mM]$$

with $k = 0, 1, \ldots, N-1, -\infty < m < \infty$. Then we have

$$x[n] = \hat{x}[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} <x, h_{k,m}> f_{k,m}[n]. \tag{8}$$

In the language of frame theory, (8) corresponds to an expansion of the input signal into $f_{k,m}[n]$ [1], [2]. It is well-known that $\{h_{k,m}\}$ is a frame if and only if PR is achieved for any $x \in \ell^2(\mathbb{R})$. So the PR condition indicates the invertibility of the analysis operator in a given set. For more details, please refer to [1], [2]. For an analysis (synthesis) FB, we will use alternatively its transfer function $H_k(z)$ ($F_k(z)$), its polyphase matrix $E(z)$ ($R(z)$) and its frame $\{h_{k,m}\}$ ($\{f_{k,m}\}$) to refer to the FB.

For an analysis filter bank with polyphase matrix $E(z)$, recall the following results from [1], [2]

Lemma 2: $\{H_k(z)\}$ implements a frame expansion if and only if its polyphase matrix $E(z)$ is of full column rank on the unit circle.

Lemma 3: An FB implements a tight frame expansion if and only if its polyphase matrix $E(z)$ is paraunitary, that is, $E(z)^{\sim}E(z) = cI$.

Corollary 1: Let $E(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be the polyphase matrix of an FB $\{H_k(z)\}$. Then $\{H_k(z)\}$ implements a frame expansion if and only if $\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}$ has full column rank for any $\theta \in [0, 2\pi)$.

Proof: It follows from Lemma 2 that $\{H_k(z)\}$ implements a frame expansion if and only if $E(e^{j\theta})$ is of full column rank for any $\theta \in [0, 2\pi)$. On the other hand, we have the following equation

$$\begin{bmatrix} A - e^{j\theta}I_n & B \\ C & D \end{bmatrix} \begin{bmatrix} (A - e^{j\theta}I_n)^{-1} & -(A - e^{j\theta}I_n)^{-1}B \\ 0 & I_M \end{bmatrix}$$
$$= \begin{bmatrix} I_n & 0 \\ C(A - e^{j\theta}I_n)^{-1} & E(e^{j\theta}) \end{bmatrix}.$$

Therefore, the column rank of $\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}$ is equal to the column rank of $E(e^{j\theta})$ plus n. This completes the proof. ■

III. Construction of the Tight Frame from an Un-tight Frame

This section provides a direct and simple method to construct a paraunitary FB from any given FB. This problem is also addressed in [1], [9]. However, the method in [1] involves spectral factorization and direct inverse of transfer matrix, which generally does not have a closed form solution and needs the approximation approach. In [9], the discussion is restricted to finite dimensional space.

Theorem 1: Given a stable analysis set $\{h_{k,m}[n]\}$, let $E(z)$ be its polyphase matrix with a minimal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Assume that $\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}$ has full column rank for all $\theta \in [0, 2\pi)$, and D has full column rank. Let $M(z)$, $N(z)$, Z, F, W and X be defined as in (1)-(5) of Lemma 1, respectively. Then

1) The frame $\{h_{k,m}[n]\}$ is a tight frame with bound α if and only if $W = \alpha I$ and $D^*C + B^*XA = 0$.

2) If $\{h_{k,m}[n]\}$ is not tight, then the frame with polyphase representation $E_t(z) = N(z)$ is tight with frame bound $\alpha = \beta = 1$, where $N(z)$ is the inner part of $E(z)$ as given in Lemma 1.

Proof: 1) It is shown in [1] that $\{h_{k,m}[n]\}$ is tight if and only if $E^{\tilde{}}(z)E(z) = \alpha I$. The result then follows directly from Lemma 21.18 (page 552 in [10]) that $E^{\tilde{}}(z)E(z) = \alpha I$ is equivalent to $W = \alpha I$ and $D^*C + B^*XA = 0$.
2) By Lemma 1, $E(z)$ can be factorized as $E(z) = N(z)M^{-1}(z)$ with $N(z)$ being inner, i.e. $N(z)^{\tilde{}}N(z) = I$. Thus the frame $E_t(z) = N(z)$ is a tight frame with frame bound equal to 1. ■

Remark 1: Theorem 1 shows that the solutions to algebraic Riccati equations and algebraic matrix manipulations are enough for computing the paraunitary FB, therefore the factorization and approximation in [1] are completely avoided. The solution of the algebraic Riccati equation (5) can be computed efficiently and reliably using software, for example, MATLAB. See MATLAB functions 'dric' for detail.

Remark 2: For any FB frame $\{h_{k,m}[n]\}$, we can always find an $E(z)$ with feedthrough term D full column rank by allowing PR with some delay. Note that PR is always achievable for FB frame.

IV. State space computation

This section discusses the complete computation procedure for using Theorem 1. In other words, given the analysis filters $H_0(z), H_1(z), \ldots, H_{N-1}(z)$, and the decimation ratio M with $M \leq N$, the procedure to obtain a paraunitary FB from non-paraunitary $H_i(z)$s will be presented in detail. Note that some steps are similar to the results in [12].

First, denote

$$H(z) = \left[\begin{array}{ccc} H_0(z) & \cdots & H_{N-1}(z) \end{array}\right]^T \tag{9}$$

and represent $H(z)$ in a causal (not necessarily stable) minimal state space realization

$$H(z) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right]. \tag{10}$$

The transfer matrix associated with its polyphase representation is then given by

$$\begin{aligned} E(z) &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{c|cccc} A_H^M & A_H B_H & A_H^2 B_H & \cdots & A_H^M B_H \\ \hline C_H & D_H & C_H B_H & \cdots & C_H A_H^{M-2} B_H \end{array}\right]. \end{aligned} \tag{11}$$

The MATLAB function 'dare' can be used to compute X, F and W in equation (3)-(5). The X, F and W thus computed can be used to obtain the state-space realization of $N(z)$ and $M(z)$ using the equation (2). By Theorem 1, the frame bounds as well as the state-space realization of $N(z)$ can be computed directly.

Algorithm 1 (Construction of the tight frame)

Step 1) Form the transfer matrix model of $H(z)$ as in (9) and obtain its state-space model using the MATLAB function 'tf2ss'.

Step 2) Obtain the state-space model of $E(z)$ using (11).

Step 3) Compute X, F and W in equation (3)-(5) for $E(z)$ and obtain the state-space realization of $N(z)$ and $M(z)$ using the equation (2).

Step 4) Obtain the filters $H_t(z)$ according to the polyphase representation of $N(z)$. Then $H_t(z)$ is a paraunitary filter bank corresponding to a tight frame.

Remark 3: To maintain the numerical robustness of the above algorithms, the minimal realization of the state-space model and transfer matrix model is required in model conversion. This can be easily carried out using the MATLAB function 'minreal'.

V. Numerical examples

This section presents some numerical examples to show the effectiveness and reliability of Algorithm 1.

Example 1 (Oversampled FB with IIR analysis filters) Consider the filter bank shown in Fig.1 with $N = 3$ and $M = 2$, where $H_0(z), H_1(z), H_2(z)$ are low-pass, band-pass and high-pass butterworth filters given respectively by $H_0(z) = \frac{0.4208z+0.4208}{z-0.1584}$, $H_1(z) = \frac{0.2452z^2-0.2452}{z^2+0.5095}$, and $H_2(z) = H_0(-z)$. The frequency responses of $H_1(z)$, $H_2(z)$ and $H_i(z)$ are shown in Fig. 1. After Steps 1)-2), we have

$$E(z) = \begin{bmatrix} \frac{0.4208z+0.06665}{z-0.02509} & \frac{0.4875z}{z-0.02509} \\ \frac{0.2452z-0.2452}{z+0.5095} & 0 \\ \frac{0.4208z+0.06665}{z-0.02509} & \frac{-0.4875z}{z-0.02509} \end{bmatrix},$$

the frame bounds are $\alpha = 0.6$, and $\beta = 1.13$. Obviously, the analysis filter bank is not paraunitary. Using Step 3) of Algorithm 1, we get

$$N(z) = \begin{bmatrix} \frac{0.5533z^2+0.3696z+0.04465}{z^2+0.3162z+0.0520} & 0.7071 \\ \frac{0.3225z^2-0.3305z+0.0081}{z^2+0.3162z+0.0520} & 0 \\ \frac{0.5533z^2+0.3696z+0.04465}{z^2+0.3162z+0.0520} & -0.7071 \end{bmatrix}$$

and the corresponding paraunitary filters

$$\begin{aligned} H_{t0}(z) &= \frac{\begin{array}{c}(0.5533z^5 + 0.7071z^4 + 0.3696z^3 \\ +0.2236z^2 + 0.04465z + 0.0368)\end{array}}{z^5 + 0.3162z^3 + 0.0520z} \\ H_{t1}(z) &= \frac{0.3225z^4 - 0.3305z^2 + 0.0081}{z^4 + 0.3162z^2 + 0.0520} \\ H_{t2}(z) &= \frac{\begin{array}{c}(0.5533z^5 - 0.7071z^4 + 0.3696z^3 \\ -0.2236z^2 + 0.04465z - 0.0368)\end{array}}{z^5 + 0.3162z^3 + 0.0520z}. \end{aligned}$$

The frequency response of $H_{t0}(z)$, $H_{t1}(z)$ and $H_{t2}(z)$ are shown in Fig. 2

Example 2 (Oversampled FB with FIR analysis filters) Consider the oversampled FB shown in Fig.1 with $N = 3$ and $M = 2$, where $H_0(z), H_1(z)$ and $H_2(z)$ are the following Parks-McClellan optimal equiripple FIR filters

$$\begin{aligned} H_0(z) &= 0.239 + 0.6655z^{-1} + 0.6655z^{-2} + 0.239z^{-3} \\ H_1(z) &= -0.5189z^{-1} + 0.6793z^{-3} - 0.5189z^{-5} \\ H_2(z) &= H_0(-z). \end{aligned}$$

The frequency response of $H_0(z), H_1(z)$ and $H_2(z)$ are shown in Fig. 3. The frame bounds calculated with algorithm 1 are

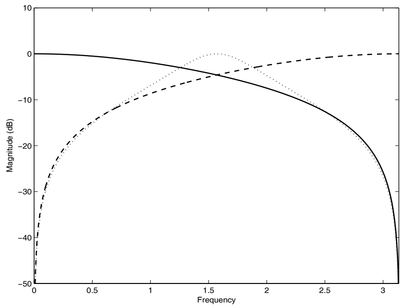

Fig. 1. Example 1: Frequency responses of the analysis filters. $H_0(z)$: solid; $H_1(z)$: dotted; $H_2(z)$: dashed

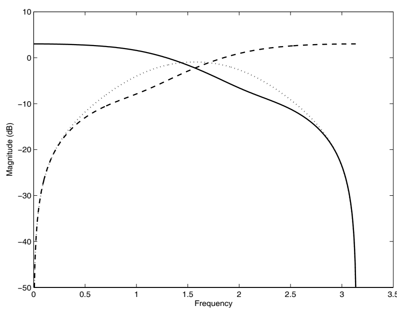

Fig. 2. Example 1: Frequency responses of the paraunitary filters. $H_{t0}(z)$: solid; $H_{t1}(z)$: dotted; $H_{t2}(z)$: dashed.

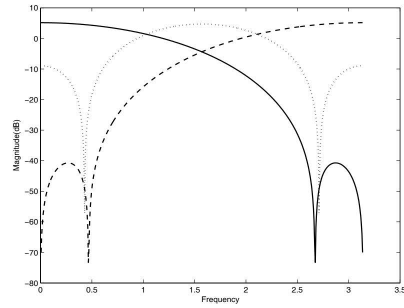

Fig. 3. Example 2: Frequency responses of the analysis filters. $H_0(z)$: solid; $H_1(z)$: dotted; $H_2(z)$: dashed

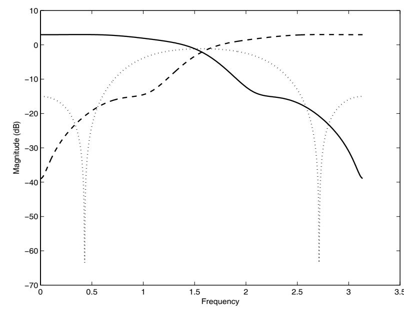

Fig. 4. Example 2: Frequency responses of the paraunitary filters. $H_{t0}(z)$: solid; $H_{t1}(z)$: dotted; $H_{t2}(z)$: dashed

$\alpha = 0.6032$ and $\beta = 1.82$. Thus $H_i(z)$ are not a tight frame. Using Algorithm 1, we get

$$N(z) = \begin{bmatrix} \frac{0.2539z+0.7071}{z+0.3591} & \frac{0.4826z^2+0.1733z}{z^2-0.1782z+0.1416} \\ 0 & \frac{-0.3763z^2+0.4926z-0.3763}{z^2-0.1782z+0.1416} \\ \frac{0.2539z+0.7071}{z+0.3591} & \frac{-0.4826z^2-0.1733z}{z^2-0.1782z+0.1416} \end{bmatrix},$$

and the paraunitary filter banks corresponding to a tight frame are as follows

$$\begin{aligned} H_{t0}(z) &= \frac{\begin{array}{c}0.2539z^6 + 0.4826z^5 + 0.6618z^4 + 0.3466z^3 \\ +0.0901z^2 + 0.06225z + 0.1001\end{array}}{z^6 + 0.1809z^4 + 0.0776z^2 + 0.05086} \\ H_{t1}(z) &= \frac{-0.3763z^4 + 0.4926z^2 - 0.3763}{z^5 - 0.1782z^3 + 0.1416z} \\ H_{t2}(z) &= H_{t0}(-z). \end{aligned}$$

The frequency response of $H_{t0}(z), H_{t1}(z)$ and $H_{t2}(z)$ are shown in Fig. 4.

As shown in these examples, no matter what type of analysis FB is given, the frame bounds and tight frames of the given analysis FB can be easily calculated using the procedure given in Algorithms 1 without approximation.

VI. Conclusion

This paper has provided explicit and numerically efficient formulae to obtain the tight oversampled FB frame from an arbitrary oversampled FB (IIR or FIR) frame. These formulae do not involve any approximation. The results provide a unified framework for the construction of tight frame generated by oversampled FBs (IIR and FIR).

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