# Least Squares Design of Orthonormal Wavelets via the Zero-Pinning Technique

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***Abstract*— A simple yet versatile technique [1] was recently introduced for designing FIR orthonormal wavelet filters. The technique involves pinning some of the zeros of the Parametric Bernstein Polynomial to ensure non-negativity of the frequency response. Filters with a high number of vanishing moments and sharper frequency response (but lower vanishing moments) than the maximally flat Daubechies filters can be easily designed. The position of the pinned zeros can be easily adjusted to give a variety of frequency response. This paper extends the previous work and presents a method to determine the zeros' position that will give a least squares error in the stopband response.**

## I. Introduction

Orthonormal wavelets are constructed from an appropriately designed conjugate quadrature filter (CQF). The CQF $H(z)$ is obtained from a spectral factorization of a product filter $P(z)$, ie. $H(z)H(z^{-1}) = P(z)$. The product filter must satisfy the following **halfband constraint**: $P(z) + P(-z) = 1$ and **non-negativity constraint**: $P(e^{j\omega}) \geq 0$. The orthonormal wavelet $\psi(t)$ (spectrum $\Psi(\omega)$) is generated from the filter $H(z)$ and is given by the **infinite product formula:** $\Psi(\omega) = \frac{1}{2} H_1\left(e^{j\,\omega/2}\right) \quad \prod_{k=1}^{\infty}\left\{\frac{1}{2} H\left(e^{j\,\omega/2^{k+1}}\right)\right\}$ where $H_1(z) = z^{-1}H(-z^{-1})$. To ensure convergence, zeros at $z = -1$ are imposed on $H(z)$ and this is also known as the vanishing moment (VM) condition [2]. Generally, the higher the number of VM, the higher the smoothness (in terms of the order of differentiability) [2].

The celebrated wavelets of Daubechies [2] are obtained by imposing a maximum number of VM on $H(z)$ (or equivalently $P(z)$). This gave a maximally flat frequency response but slow transition band roll-off. Sharper roll-off filters require ripples in the stopband (and/or passband). The opposite to the maximally flat response is the equiripple response. The equiripple CQF of Smith and Barnwell [3] does not have any VM; hence cannot produce wavelets. Rioul and Duhamel [4] proposed a Remez exchange algorithm to design equiripple CQF filters with prescribed VM and to satisfy the non-negativity constraint. A simple alternative to the Remez based algorithm was recently proposed in [1] based on the concept of zero-pinning the Bernstein Polynomial. This technique ensured that the non-negativity constraint and the VM condition are satisfied simultaneously.

The zero-pinning technique requires the specification of the zeros' position. Once this is done it is then a simple matter of solving linear equations to complete the design of the product filter. Heuristic strategies for determining the positions were presented in [1]. In this paper we present a more objective approach to determining the zeros' position. A least squares formulation is adopted and the derivation of the design equations together with examples will be presented.

## II. Zero-Pinning Technique

We very briefly describe the zero-pinning technique and the Parametric Bernstein Polynomial (PBP) and refer the reader to [1] for more details. The PBP was first introduced by Caglar and Akansu [5] and is given by:

$$B_N(x;\alpha) \equiv \sum_{i=0}^{N} f(i) \begin{pmatrix} N \\ i \end{pmatrix} x^i (1-x)^{N-i} \tag{1}$$

where $N$ is odd, $\alpha = [\ \alpha_0 \quad \ldots \quad \alpha_{(N-1)/2}\ ]^T$ and

$$f(i) \equiv \begin{cases} 1-\alpha_i & 0 \leq i \leq \frac{1}{2}(N-1) \\ \alpha_{N-i} & \frac{1}{2}(N+1) \leq i \leq N \end{cases} \tag{2}$$

The PBP can also be expressed as:

$$B(x) = K(x) - \sum_{l=L+1}^{(N-1)/2} k_l(x)\,\alpha_l \tag{3}$$

where

$$K(x) \equiv \sum_{i=0}^{(N-1)/2} \begin{pmatrix} N \\ i \end{pmatrix} x^i (1-x)^{N-i}$$

$$k_l(x) \equiv \begin{pmatrix} N \\ l \end{pmatrix} [x^l (1-x)^{N-l} - x^{N-l}(1-x)^l].$$

The polynomial satisfies the *halfband condition*: $B(x)+B(1-x) = 1$. The product filter $P(z)$ of the CQF can be obtained by: $P(z) = B(-\frac{1}{4}z(1-z^{-1})^2)$. The desired number of zeros at $z = -1$ of $P(z)$ can be imposed by setting an appropriate number of Bernstein parameters to zero. Specifically setting $\alpha_i = 0$ for $i = 0, \ldots, L$ will give $2(L+1)$ zeros; hence $(L+1)$ zeros for the CQF $H(z)$.

The halfband and VM conditions are structurally imposed and this is the appeal of the PBP for wavelet filter design. However, there is still the non-negative condition that needs to be satisfied before the PBP can be used for orthogonal filter design, ie. $B(x) \geq 0$ for $0 \leq x \leq 1$. This is where the zero-pinning technique comes into the picture and it basically amounts to explicitly pinning the "down ripples" (local minima) to the zero-axis:

$$B(x_i) \;=\; K(x_i) - \sum_{l=L+1}^{(N-1)/2} k_l(x_i)\,\alpha_l \;\;=\;\; 0 \qquad (4)$$

$$B'(x_i) \;=\; K'(x_i) - \sum_{l=L+1}^{(N-1)/2} k'_l(x_i)\,\alpha_l \;\;=\;\; 0 \qquad (5)$$

where $x_1, x_2, \ldots, x_P$ are the zero locations for pinning. The $x_i$'s are all in the stopband, ie. $1/2 < x_i < 1$. Each $x_i$ contributes two linear equations in the non-zero parameters $\alpha_l$'s. The value $L$ should be chosen such that there is an even number of non-zero $\alpha_l$'s, ie. $(N-1)/2 - L$ must be even. The number of pinned zeros are then $P = \frac{1}{2}((N-1)/2 - L)$. There will be $2P$ linear equations with $2P$ unknowns which can be solved easily to give the $\alpha_l$ for $l = (L+1), \ldots, (N-1)/2$.

*Non-negativity of polynomial*

The fact that the pinned zeros are local minima, thus ensuring non-negativity can be argued as follows. The halfband condition implies that $B'(x) = B'(1-x)$, ie. the derivative is symmetric about $x = 1/2$. Using the fact that (i) $B(x)$ has a $(1-x)^{L+1}$ factor; and (ii) derivative symmetry, it can be easily shown that

$$B'(x) = x^L (1-x)^L Q(x)$$

where $Q(x)$ is a degree $(N-1-2L)$ polynomial. The zeros of $Q(x)$ are the location of the local optima (minima and maxima). Between any (i) two adjacent pinned zeros; or (ii) the last pinned zero and $x = 1$, there must be (at least) one local optimum. Therefore there are (at least) the same number of un-pinned optima as there are pinned optima. Half of the optima are in the passband ($0 < x < 1/2$) and half are in the stopband ($1/2 < x < 1$) due to symmetry. Adding all up gives exactly $4P = (N-1-2L)$ optima which is the same as the degree of $Q(x)$, ie. all optima are accounted for. Now if the first pinned zero $x_1$ is a local maxima (instead of minima), there must be (at least) one local minima between $x = 1/2$ (noting that $B(1/2) = 1/2$) and $x = x_1$ but this is a violation since all optima have been accounted for. Continuing with a similar argument to other pinned zeros will show that the zeros must be local minima.

In essence, by using all the degrees of freedom (in the non-zero Bernstein parameters) to pin the zeros to the horizontal axis, non-negative local minima are created thus ensuring non-negativity of the polynomial. More details about the technique including the motivation for its development are found in [1].

## III. Least Squares Design

Equations (4) and (5) implicitly define the $\alpha$ values in terms of the zeros $x_i$, ie. we have the following functional relationship:

$$\alpha_l = \alpha_l(x_1, x_2, \ldots, x_P) \qquad (6)$$

for $l = (L+1), \ldots, (N-1)/2$. Consider the following objective function: $\tilde{E}_S \equiv \int_{\omega_s}^{\pi} |H(e^{j\omega})|^2\, d\omega$ where $\omega_s$ is the stopband edge and $\pi/2 < \omega_s < \pi$. Now $\tilde{E}_S$ is a measure of the CQF's stopband energy and it can also be expressed in terms of the PBP as follows: $\tilde{E}_S = \int_{\omega_s}^{\pi} B(\sin^2(\omega/2))\, d\omega$ since $P(z) = H(z)H(z^{-1}) = B(-\frac{1}{4}z(1-z^{-1})^2)$. Instead of using $\tilde{E}_S$ which involves integrating trigonometric functions, we consider a related objective function given by

$$E_S \equiv \int_{x_S}^{1} B(x)\, dx \qquad (7)$$

where $x_S = \sin^2(\omega_s/2)$. $E_S$ is simpler to compute than $\tilde{E}_S$ as the integrand in $E_S$ is easily obtained whereas the integrand in $\tilde{E}_S$ needs to be manipulated using trigonometric identities before integration can be performed. Using $E_S$ is equivalent to using the weighting function $W(\omega) = \frac{1}{2}\sin\omega$ (as $dx = \frac{1}{2}\sin\omega\, d\omega$). This will give small weight values to frequencies in the vicinity of $\omega = \pi$. However this will not be significant as the attenuation of $H(e^{j\omega})$ will be high in that vicinity due to the structurally imposed zeros at $z = -1$. If desired, a polynomial weighting function $W(x)$ that approximates the inverse weighting, ie. $W(x) \approx (x(1-x))^{-1/2}$ (using a Taylor Series expansion), can be applied to (7). The use of a polynomial function $W(x)$ will ensure that the integration in (7) can be performed analytically as the integrand $B(x)W(x)$ is a polynomial.

Using (3) in (7) we have:

$$E_S = a_0 + \sum_{l=1}^{2P} a_l\, \alpha_{L+l} \qquad (8)$$

where the constants $a_0, a_1, \ldots, a_{2P}$ are given by: $a_0 \equiv \int_{x_S}^{1} K(x)\, dx$ and $a_l \equiv \int_{x_S}^{1} k_l(x)\, dx$ for $l = 1, \ldots, 2P$. Since the $\alpha_l$'s are functionally dependent on the zeros $x_i$ as shown in (6), $E_S$ will also have the same functional dependence, ie. $E_S = E_S(x_1, x_2, \ldots, x_P)$. Theoretically, if the functions in (6) are explicitly obtained, they can then be substituted into (8) to give an explicit function of $E_S(x_1, x_2, \ldots, x_P)$. Minimizing $E_S$ w.r.t. to $x_i$ requires the solution of the equations $\frac{\partial E_S}{\partial x_i} = 0$ for $i = 1, \ldots, 2P$. However, solving equations (4) and (5) for $\alpha_l$'s explicitly with general values of $x_i$ (ie. not numerical values) is practically intractable. Even with one pinned zero, using this explicit technique results in a monstrous expression for $E_S(x_1)$.

An implicit technique to minimizing $E_S$ is needed which is presented next. The key idea in the technique is to exploit the implicit functional dependence embedded in (4) and (5) by performing implicit differentiation of these equations. The technique is best explained by first considering the simplest case with one pinned zero. We then consider the case with two pinned zeros before generalizing to an arbitrary number of zeros. For brevity of notation the following symbols shall be used: $\tilde{\alpha}_l \equiv \alpha_{L+l}$ and $\tilde{k}_l \equiv k_{L+l}$ for $l = 1, \ldots, 2P$. We refer to (4) and (5) with a specific $i$ value (ie. specific zero $x_i$) as (4)-i and (5)-i respectively, eg. (4)-1 is equation (4) with $x_i = x_1$.

### A. One Pinned Zero $x_1$

Differentiating (8) w.r.t. $x_1$ and equating to zero gives: $\frac{dE_S}{dx_1} = a_1 \frac{d\tilde{\alpha}_1}{dx_1} + a_2 \frac{d\tilde{\alpha}_2}{dx_1} = 0$. Differentiating (4)-1 w.r.t. $x_1$ and using (5)-1 gives: $\tilde{k}_1(x_1)\frac{d\tilde{\alpha}_1}{dx_1} + \tilde{k}_2(x_1)\frac{d\tilde{\alpha}_2}{dx_1} = 0$. In matrix form we have:

$$\begin{bmatrix} a_1 & a_2 \\ \tilde{k}_1(x_1) & \tilde{k}_2(x_1) \end{bmatrix} \begin{bmatrix} \frac{d\tilde{\alpha}_1}{dx_1} \\ \frac{d\tilde{\alpha}_2}{dx_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With $\frac{d\tilde{\alpha}_1}{dx_1}$ and $\frac{d\tilde{\alpha}_2}{dx_1}$ considered as unknowns, the equations are simultaneous homogeneous equations. The trivial solution (all zeros) is one possible solution to homogeneous equations but this would mean (i) $\frac{d\tilde{\alpha}_1}{dx_1} = 0$ and (ii) $\frac{d\tilde{\alpha}_2}{dx_1} = 0$. In general, the solution (for the $x_1$ value) to (i) will be different to the solution to (ii). For non-trivial solutions, the determinant must be zero, ie. $\begin{vmatrix} a_1 & a_2 \\ \tilde{k}_1(x_1) & \tilde{k}_2(x_1) \end{vmatrix} = 0$. Solving the determinant equation gives the optimal value of $x_1$.

### B. Two Pinned Zeros $x_1$, $x_2$

Differentiating (8) w.r.t. $x_1$ and equating to zero gives: $a_1 \frac{\partial\tilde{\alpha}_1}{\partial x_1} + a_2 \frac{\partial\tilde{\alpha}_2}{\partial x_1} + a_3 \frac{\partial\tilde{\alpha}_3}{\partial x_1} + a_4 \frac{\partial\tilde{\alpha}_4}{\partial x_1} = 0$. Differentiating (4)-1 w.r.t. $x_1$ and using (5)-1 gives: $\tilde{k}_1(x_1)\frac{\partial\tilde{\alpha}_1}{\partial x_1} + \tilde{k}_2(x_1)\frac{\partial\tilde{\alpha}_2}{\partial x_1} + \tilde{k}_3(x_1)\frac{\partial\tilde{\alpha}_3}{\partial x_1} + \tilde{k}_4(x_1)\frac{\partial\tilde{\alpha}_4}{\partial x_1} = 0$. Differentiating (4)-2 and (5)-2 w.r.t. $x_1$ give $\tilde{k}_1(x_2)\frac{\partial\tilde{\alpha}_1}{\partial x_1} + \tilde{k}_2(x_2)\frac{\partial\tilde{\alpha}_2}{\partial x_1} + \tilde{k}_3(x_2)\ \frac{\partial\tilde{\alpha}_3}{\partial x_1} + \tilde{k}_4(x_2)\frac{\partial\tilde{\alpha}_4}{\partial x_1} = 0$ and $\tilde{k}'_1(x_2)\ \frac{\partial\tilde{\alpha}_1}{\partial x_1} + \tilde{k}'_2(x_2)\ \frac{\partial\tilde{\alpha}_2}{\partial x_1} + \tilde{k}'_3(x_2)\ \frac{\partial\tilde{\alpha}_3}{\partial x_1} + \tilde{k}'_4(x_2)\ \frac{\partial\tilde{\alpha}_4}{\partial x_1} = 0$ respectively. Putting the four equations in matrix form, we have:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \tilde{k}_1(x_1) & \tilde{k}_2(x_1) & \tilde{k}_3(x_1) & \tilde{k}_4(x_1) \\ \tilde{k}_1(x_2) & \tilde{k}_2(x_2) & \tilde{k}_3(x_2) & \tilde{k}_4(x_2) \\ \tilde{k}'_1(x_2) & \tilde{k}'_2(x_2) & \tilde{k}'_3(x_2) & \tilde{k}'_4(x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial\tilde{\alpha}_1}{\partial x_1} \\ \frac{\partial\tilde{\alpha}_2}{\partial x_1} \\ \frac{\partial\tilde{\alpha}_3}{\partial x_1} \\ \frac{\partial\tilde{\alpha}_4}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

which are simultaneous homogeneous equations. Denoting the $4\times4$ matrix as $\mathbf{K}(x_1, x_2)$, the condition for non-trivial solution is the following determinant equation:

$$|\mathbf{K}(x_1, x_2)| = 0 \tag{10}$$

We repeat the above process but with the roles of $x_1$ and $x_2$ interchanged. All differentiation will be w.r.t. $x_2$. To get the second equation, (4)-2 and (5)-2 are utilized. To get the third and fourth equations, (4)-1 and (5)-1 are utilized. A matrix equation similar to (9) is obtained but with the following differences: (i) The unknowns are derivatives w.r.t. $x_2$, eg. $\frac{\partial\tilde{\alpha}_1}{\partial x_2}$ etc. ; (ii) The matrix of coefficient is $\mathbf{K}(x_2, x_1)$ (ie. $x_1$ and $x_2$ are interchanged). Applying the condition for non-trivial solution gives the following determinant equation:

$$|\mathbf{K}(x_2, x_1)| = 0 \tag{11}$$

Solving the determinant equations (10) and (11) simultaneously gives the optimal value of $x_1$ and $x_2$.

### C. General Case

With $P$ pinned zeros there will be $P$ determinant equations which can be solved simultaneously to give the $P$ pinned zeros. To get the first determinant $\mathbf{K}(x_1, x_2, \ldots, x_P)$, we apply the generalization of the process described above:

1) Differentiate (8) w.r.t. $x_1$ and equate to zero to get one equation.
2) Differentiate (4)-1 w.r.t. $x_1$ and use (5)-1 to get another equation.
3) Differentiate (4)-1 and (5)-1 w.r.t. $x_p$, $p \neq 1$, one $p$ at a time for all $p$ ($\neq 1$), to give the remaining $2P - 2$ equations.
4) Putting all the equations together in matrix form and equating the determinant of the matrix (of size $2P\times2P$) to zero gives the determinant equation.

Using symmetry arguments, the other determinants can be obtained by interchanging $x_1$ with $x_l$ ($l \neq 1$) in the first determinant function $\mathbf{K}(x_1, x_2, \ldots, x_P)$.

### D. Solving the Determinant Equations

The solution of simultaneous (non-linear) polynomial equations is required to determine the optimal zeros. There are a plethora of methods for solving non-linear system of equations but a relatively straightforward method is multidimensional version of the classical Newton-Raphson method [6]. We utilized this method and found it sufficient for the task at hand. The method can be described as follows. Suppose the equation (vector form) to be solved is given by $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. Starting with an initial solution $\mathbf{x}_0$, the solution is successively updated as $\mathbf{x}_{new} = \mathbf{x}_{old} + \delta\mathbf{x}$. The correction vector $\delta\mathbf{x}$ is obtained by solving the linear equation $\mathbf{J}.\delta\mathbf{x} = -\mathbf{F}$ where $\mathbf{J}$ is the Jacobian of $\mathbf{F}$ (ie. $J_{i,j} = \frac{\partial F_i}{\partial x_j}$) and both $\mathbf{F}$ and $\mathbf{J}$ are evaluated at $\mathbf{x} = \mathbf{x}_{old}$. The iterative process is terminated when $||\mathbf{x}_{new} - \mathbf{x}_{old}||$ or $||\mathbf{F}(\mathbf{x}_{new})||$ is sufficiently small.

## IV. Design Examples

Several examples will be presented to illustrate the versatility of the technique. In all examples the response of the maximally flat filter (polynomial) with the same length (as the example) will also be shown. A comparison that will show increased sharpness with zero pinning can then be readily made.

*Example 1:* The values $N = 15$ and $L = 5$ will yield a length 16 CQF with 6 VM (zeros at $z = -1$). There are 2 non-zero $\alpha_l$'s allowing one pinned zero. Choose $x_S = 0.7$ (stopband edge). For this simple case the Newton-Raphson method is not needed and the determinant equation reduces to a quadratic equation given by: $8x_1^2 - 8x_1 + 7(1 - x_S)x_S = 0$. The optimal zero is $x_1 = 0.7574$ resulting in $(\alpha_6, \alpha_7) = (0.4950, -2.3253)$. The frequency response (in the variable $x$) of the product filter is shown in figure 1 (solid line) where the maximally flat equivalent (dashed line) is also shown.

*Example 2:* Same as example 1 except now $x_S = 0.6$ (stopband edge). The optimal zero is $x_1 = 0.7$ resulting in $(\alpha_6, \alpha_7) = (1.5607, -5.6452)$. The frequency response (in the variable $x$) of the product filter is shown in figure 1 (dotted line) where the maximally flat equivalent (dashed line) is also shown.

The examples above illustrate the trade-off mechanism (between sharpness and ripple size). Note that the first pinned zero $x_1 > x_S$ and this also occurs for other examples with more pinned zeros.

*Example 3:* The values $N = 15$ and $L = 3$ will yield a length 16 CQF with 4 VM (zeros at $z = -1$). There are 4 non-zero $\alpha_l$'s allowing two pinned zeros. Choose $x_S = 0.65$ (stopband edge). Using the Newton-Raphson method, it took less than 10 iterations to converge to the following solution: $(x_1, x_2) = (0.7053, 0.8415)$ resulting in $(\alpha_4, \alpha_5, \alpha_6, \alpha_7) = (0.3923, -2.8048, 9.6177, -18.7722)$. The frequency response (in the variable $x$) of the product filter is shown in figure 2 (solid line) where the maximally flat equivalent (dashed line) is also shown. Compared with example 2 there is reduced ripple magnitude but also reduced flatness at $\omega = \pi$, ie. reduced VM (note that the transition band sharpness is about the same in both examples).

*Example 4:* The values $N = 19$ and $L = 3$ will yield a length 20 CQF with 4 VM (zeros at $z = -1$). There are 6 non-zero $\alpha_l$'s allowing three pinned zeros. Choose $x_S = 0.6$ (stopband edge). Using the Newton-Raphson method, it took less than 10 iterations to converge to the following solution: $(x_1, x_2, x_3) = (0.6545, 0.7790, 0.8946)$ resulting in $(\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9) = (0.5409, -5.1812, 25.1034, -77.6745, 164.868, -243.017)$. The frequency response (in the variable $x$) of the product filter is shown in figure 3 (solid line) where the maximally flat equivalent (dashed line) is also shown. This example shows that filter with sharp transition roll-off can be designed with this technique.

In the examples above the stopband response values are close to 0 and one may argue that zero-pinning may not be necessary. Our experience shows that minimzing the stopband energy of the PBP without any constraint to ensure non-negativity will result in product filters with negative response values (see [1] for discussion). This will create problems during the spectral factorization process. This approach is however viable if the PBP is used instead for biorthogonal filter design [7], [8] as non-negativity is not mandatory there.

## V. Summary

A least squares approach has been presented for the design of orthonormal wavelet filters with a prescribed number of vanishing moments. The technique utilizes the principle of zero-pinning the Parametric Bernstein Polynomial. The position of the pinned zeros were chosen to minimize a stopband energy measure of the filter. An implicit differentiation technique was presented to derive the equations governing the position of the optimal zeros. The design process requires the solution of a system of non-linear (polynomial) equations which was achieved through the use of the multidimensional Newton-Raphson algorithm. Examples were presented to demonstrate the flexibility of the technique for designing filters with a variety of characteristics.

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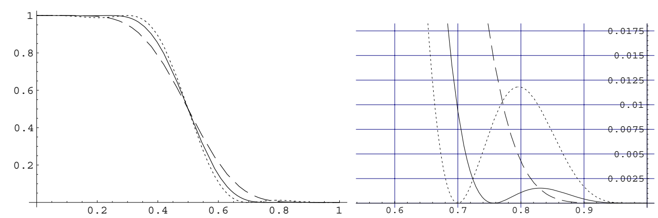

Fig. 1. Degree 15 Parametric Bernstein Polynomial. Solid line: example 1. Dotted line: example 2. Dashed line: maximally flat.

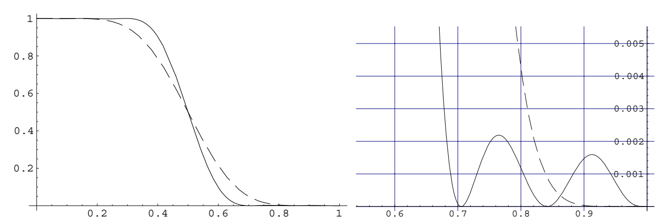

Fig. 2. Degree 15 Parametric Bernstein Polynomial. Solid line: example 3. Dashed line: maximally flat.

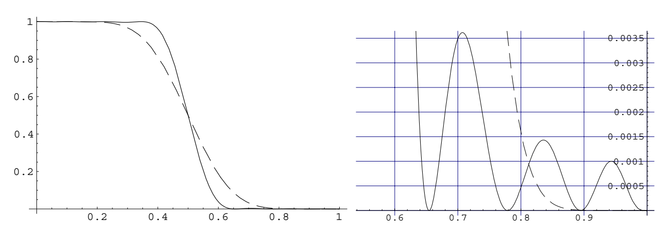

Fig. 3. Degree 19 Parametric Bernstein Polynomial. Solid line: example 3. Dashed line: maximally flat.