ANALYSIS OF A MULTICHANNEL FILTERED-X SET-MEMBERSHIP AFFINE PROJECTION ALGORITHM

Alberto Carini

STI - University of Urbino
Piazza della Repubblica, 13 - 61029 Urbino, Italy
email: carini@sti.uniurb.it

Giovanni L. Sicuranza

DEEI - University of Trieste
Via A. Valerio, 10 - 34127 Trieste, Italy
email: sicuranza@univ.trieste.it

ABSTRACT

The paper provides an analysis of transient and steady-state behavior of a multichannel filtered-x set-membership affine projection algorithm, suitable for multichannel active noise control. The analysis relies on energy conservation arguments and it does not apply the independence theory nor it imposes any restriction to the signal distributions. The analysis results show that the filtered-x set-membership affine projection algorithm can reduce the computational complexity without trading residual mean-square-error and convergence speed.

1. INTRODUCTION

The choice of an adaptation algorithm for multichannel active noise controllers is still a challenge for the DSP engineer. The major concerns are the correlation between the outputs of the different channels and the heavy computational needs caused by the long impulse responses of the acoustic paths and by the high number of reference microphones, actuators, and error microphones. As a matter of fact, the common multichannel filtered-x LMS algorithm often shows very poor convergence speed in the presence of strong correlation between the error microphone signals. Better convergence behavior can be obtained by using filtered-x affine projection (Fx-AP) algorithms, but the computational needs may significantly increase. Recently, filtered-x set-membership affine projection (Fx-SM-AP) algorithms have been proposed in [1] in order to attain the same improved convergence behavior of Fx-AP algorithms but with reduced computational complexity. Indeed, the set-membership (SM) criterion does not trade computational complexity with convergence speed or with residual mean-square-error as with most adaptation algorithms.

Very few results can be found in the literature dealing with the analysis of filtered-x, affine projection or set-membership algorithms. The convergence analysis results for these algorithms are often based on the independence theory (IT) and they constrain the probability distribution of the input signal to be Gaussian or spherically invariant [2]. The IT hypothesis assumes statistical independence of time-lagged input data vectors. As it is too strong for filtered-x [3] and AP algorithms [4], different approaches have been studied in the literature in order to overcome this assumption. In [3] an analysis of the mean weight behavior of the filtered-x LMS algorithm is presented based only on neglecting the correlation between coefficient and signal vectors. Moreover, the analysis of [3] does not impose any restriction on the signal distributions. Another analysis approach that avoids IT is applied in [4] for the mean-square performance analysis of AP algorithms. This relies on energy conservation arguments and no restriction is imposed on the signal distributions. In [5], we applied and adapted the approach of [4] for analyzing the convergence behavior of multichannel Fx-AP algorithms. In this paper we extend the analysis approach of [5] and study the transient and steady-state behavior of a Fx-SM-AP algorithm. The analysis results confirm that the Fx-SM-AP algorithm can reduce the computational complexity without trading residual mean-square-error or convergence speed.

The paper is organized as follows. Section 2 reviews the multichannel feedforward active noise controller structure and introduces the Fx-SM-AP algorithm. Section 3 discusses the asymptotic solution of the Fx-SM-AP algorithm and compares it with that of Fx-AP algorithms and with the minimum-mean-square solution of the ANC problem. Section 4 presents an analysis of the transient and steady-state behavior of the Fx-SM-AP algorithm. Section 5 provides some experimental results. Conclusions follow in Section 6.

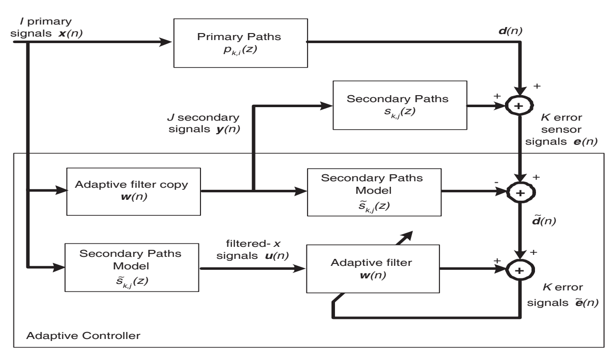


Fig. 1. Delay-compensated filtered-x structure for active noise control.

This work was supported by the MIUR under Grant PRIN 2004092314.

Throughout this paper, small boldface letters are used to denote vectors and bold capital letters are used to denote matrices, e.g., $\mathbf{x}$ and $\mathbf{X}$, all vectors are column vectors, the boldface symbol $\mathbf{I}$ indicates an identity matrix of appropriate dimensions, the symbol $\odot$ denotes linear convolution, diag$\{\ldots\}$ is a block-diagonal matrix of the entries, $E[\cdot]$ denotes mathematical expectation, $\|\cdot\|_{\mathbf{\Sigma}}$ is the weighted Euclidean norm, e.g., $\|\mathbf{w}\|_{\mathbf{\Sigma}} = \mathbf{w}^T\mathbf{\Sigma}\mathbf{w}$ with $\mathbf{\Sigma}$ a symmetric positive definite matrix, sign$(\cdot)$ is the sign function, vec$\{\cdot\}$ indicates the vector operator and vec$^{-1}\{\cdot\}$ the inverse vector operator that returns a square matrix from an input vector of appropriate dimensions, $\otimes$ denotes the Kronecker product, Prob$\{A\}$ is the probability of the event A.

2. FILTERED-X SET-MEMBERSHIP AFFINE PROJECTION ALGORITHM

Active noise controllers are based on the destructive interference in given locations of the noise produced by some primary sources and the interfering signals generated by some secondary sources. Fig. 1 shows the block diagram of a multichannel delay-compensated filtered-x active noise control system. As usual, the primary and secondary paths, which propagate the primary and secondary source signals, respectively, are modelled with linear FIR filters. In order to compensate for the propagation delay introduced by the secondary paths, the outputs $\mathbf{d}(n)$ of the primary paths are estimated by subtracting the outputs of the secondary path models from the error sensors signals $\mathbf{e}(n)$. In this paper we assume perfect modelling of the secondary paths [we consider $\tilde{s}_{k,j}(z) = s_{k,j}(z)$ for any choice of j and k], but this limitation can be easily removed by following the same methodology of [5].

For simplicity, we assume that any input i of the adaptive controller is connected to any output j with an FIR filter. It is worth noting that the theory we present in Sections 3 and 4 can be applied to any linear or nonlinear filter whose output depends linearly on the filter coefficients [5].

The following notation is used throughout the paper:
I, J, and K are the number of primary source signals, secondary source signals, and error sensors, respectively,
L is the affine projection order,
$s_{k,j}(n)$ is the impulse response of the secondary path that connects the jth secondary source to the kth error sensor,
$\mathbf{w}_{j,i}(n)$ is the coefficient vector of the FIR filter that connects the input i to the output j of the adaptive controller,
$\mathbf{x}_i(n)$ is the ith primary source input signal vector,
$\mathbf{x}(n) = \big[\mathbf{x}_1^T(n), \ldots, \mathbf{x}_I^T(n)\big]^T$,
$\mathbf{w}_j(n) = \big[\mathbf{w}_{j,1}^T(n), \ldots, \mathbf{w}_{j,I}^T(n)\big]^T$,
$y_j(n) = \mathbf{w}_j^T(n)\mathbf{x}(n)$ is the jth secondary source signal,
$d_k(n)$ is the output of the kth primary path,
$\mathbf{w}(n) = \big[\mathbf{w}_1^T(n), \ldots, \mathbf{w}_J^T(n)\big]^T$,
M is the total number of coefficients of $\mathbf{w}(n)$,
$\mathbf{u}_k(n) = \big[s_{k,1}(n) \odot \mathbf{x}^T(n), \ldots, s_{k,J}(n) \odot \mathbf{x}^T(n)\big]^T$,
$\mathbf{d}_k(n) = \big[d_k(n), \ldots, d_k(n-L+1)\big]^T$,
$\mathbf{d}(n) = \big[\mathbf{d}_1^T(n), \ldots, \mathbf{d}_K^T(n)\big]^T$,
$\mathbf{U}_k(n) = \big[\mathbf{u}_k(n), \ldots, \mathbf{u}_k(n-L+1)\big]$,
$\mathbf{U}(n) = \big[\mathbf{U}_1(n), \ldots, \mathbf{U}_K(n)\big]$,
$e_k(n) = d_k(n) + \mathbf{u}_k^T(n)\mathbf{w}(n)$,
$\mathbf{e}_k(n) = \mathbf{d}_k(n) + \mathbf{U}_k^T(n)\mathbf{w}(n)$.

The SM approach specifies an upper bound γ on the magnitude of the estimation errors $e_k(n)$. The bound γ is a design parameter that quantifies the maximum level of residual noise that can be accepted. The aim of the SM approach is to estimate the set of coefficient vectors Θ, called the *feasibility set*, given by (1),

$$\Theta = \Big\{\mathbf{w} \in \mathbb{R}^M : |d_k(n) + \mathbf{u}_k^T(n)\mathbf{w}| < \gamma \quad \forall\, k, n\Big\}. \quad (1)$$

For a properly chosen value of γ, the set Θ can comprise several valid estimates of $\mathbf{w}$. SM adaptive filters try to estimate a solution belonging to the feasibility set Θ by projecting at each time instant n the coefficient vector $\mathbf{w}(n)$ in the constraint set $\mathcal{H}_n$ given by (2),

$$\mathcal{H}_n = \Big\{\mathbf{w} \in \mathbb{R}^M : |d_k(n) + \mathbf{u}_k^T(n)\mathbf{w}| < \gamma \quad \forall\, k\Big\}. \quad (2)$$

The Fx-SM-AP algorithm considered in this paper is characterized by the adaptation rule

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \mu \sum_{k=1}^{K} \mathbf{U}_k(n)\big[\mathbf{U}_k^T(n)\mathbf{U}_k(n) + \delta\mathbf{I}\big]^{-1} \cdot \\ \cdot \frac{\text{sign}[|e_k(n)| - \gamma] + 1}{2} \cdot \mathbf{e}_k(n). \quad (3)$$

By manipulating (3), the adaptation rule can also be written in the compact form of (4), which will be used for the algorithm analysis,

$$\mathbf{w}(n+1) = \mathbf{V}(n)\mathbf{w}(n) - \mathbf{v}(n), \quad (4)$$

with $\mathbf{V}(n) = \mathbf{I} - \mu\mathbf{U}(n)\mathbf{D}(n)\mathbf{U}^T(n)$, $\mathbf{v}(n) = \mu\mathbf{U}(n)\mathbf{D}(n)\mathbf{d}(n)$,
$\mathbf{D}(n) = \text{diag}\big\{\big[\mathbf{U}_1^T(n)\mathbf{U}_1(n) + \delta\mathbf{I}\big]^{-1} f_1(n), \ldots$
$\ldots \big[\mathbf{U}_K^T(n)\mathbf{U}_K(n) + \delta\mathbf{I}\big]^{-1} f_K(n)\big\}$
and $f_k(n) = \big[\text{sign}(|e_k(n)| - \gamma) + 1\big]/2$.

3. ASYMPTOTIC SOLUTION

When the algorithm is convergent, the coefficient vector in (4) will tend for $n \to +\infty$ to a unique asymptotic vector $\mathbf{w}_\infty$, provided that the following condition is met

$$\text{Prob}\big\{|d_k(n) + \mathbf{u}_k^T(n)\mathbf{w}_\infty| > \gamma\big\} > 0. \quad (5)$$

Since active noise controllers in most cases do not admit an exact solution, for values of γ reasonably lower than the mean absolute value of $d_k(n)$ the condition in (5) is always met. If we take the expectation of (4) and consider the fixed-point of this equation, it can be easily deduced that the Fx-SM-AP algorithm, when converging, tends asymptotically to the fixed-point of the following nonlinear equation

$$\mathbf{w}_\infty = -E\Big[\mathbf{U}(n)\mathbf{D}(n)\mathbf{U}^T(n)\Big]^{-1} E\Big[\mathbf{U}(n)\mathbf{D}(n)\mathbf{d}(n)\Big], \quad (6)$$

where it should be noted that the matrix $\mathbf{D}(n)$ depends on $\mathbf{w}_\infty$ by means of $f_k(n)$ and $e_k(n)$. According to (6), the asymptotic solution $\mathbf{w}_\infty$ is independent of the step-size μ. As we already observed for Fx-AP algorithms [5], the asymptotic solution differs from the minimum-mean-square solution of the active noise control problem and it depends on the statistical properties of the input signals. Moreover, the asymptotic solution of the Fx-SM-AP algorithm also changes when varying the SM bound γ. Therefore, $\mathbf{w}_\infty$ also differs from the asymptotic solution of the Fx-AP algorithm given by (3) when $\gamma = 0$. Nevertheless, for choices of γ reasonably lower than the mean absolute value of $d_k(n)$, the experimental evidence shows that $\mathbf{w}_\infty$ remains close to the asymptotic solution of the corresponding Fx-AP algorithm.

4. TRANSIENT AND STEADY-STATE ANALYSIS

The aim of the transient analysis is to study the time evolution of the expectation of the weighted Euclidean norm of the coefficient vector $E\left[\|\mathbf{w}(n)\|^2_{\mathbf{\Sigma}}\right] = \mathbf{w}(n)^T\mathbf{\Sigma}\mathbf{w}(n)$ for some choices of the symmetric positive definite matrix $\mathbf{\Sigma}$.

By applying an approach similar to [5], the following result can be proven, which describes the transient behavior of the Fx-SM-AP algorithm.

Theorem 1 *Under the assumption that* $\mathbf{w}(n)$ *is uncorrelated with* $\mathbf{V}(n)$ *and with* $\mathbf{q}_{\mathbf{\Sigma}}(n) = \mathbf{V}^T(n)\,\mathbf{\Sigma}\,\mathbf{v}(n)$*, the transient behavior of the Fx-SM-AP algorithms with updating rule given by (3) is described by the state recursions*

$$E\left[\mathbf{w}(n+1)\right] = E\left[\mathbf{V}(n)\right]\, E\left[\mathbf{w}(n)\right] - E\left[\mathbf{v}(n)\right]$$

and $$\boldsymbol{\mathcal{W}}(n+1) = \boldsymbol{\mathcal{F}}_n\,\boldsymbol{\mathcal{W}}(n) + \boldsymbol{\mathcal{Y}}(n),$$

where

$$\boldsymbol{\mathcal{F}}_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -p_0 & -p_1 & -p_2 & \dots & -p_{M^2-1} \end{bmatrix},$$

$$\boldsymbol{\mathcal{W}}(n) = \begin{bmatrix} E[\|\mathbf{w}(n)\|^2_{\text{vec}^{-1}\{\boldsymbol{\sigma}\}} \\ E[\|\mathbf{w}(n)\|^2_{\text{vec}^{-1}\{\mathbf{F}_n\boldsymbol{\sigma}\}} \\ \vdots \\ E[\|\mathbf{w}(n)\|^2_{\text{vec}^{-1}\{\mathbf{F}_n^{M^2-1}\boldsymbol{\sigma}\}} \end{bmatrix},$$

$$\boldsymbol{\mathcal{Y}}(n) = \begin{bmatrix} \left\{\mathbf{g}_n^T + 2\,E[\mathbf{w}^T(n)]\mathbf{Q}_n\right\}\boldsymbol{\sigma} \\ \left\{\mathbf{g}_n^T + 2\,E[\mathbf{w}^T(n)]\mathbf{Q}_n\right\}\mathbf{F}_n\boldsymbol{\sigma} \\ \vdots \\ \left\{\mathbf{g}_n^T + 2\,E[\mathbf{w}^T(n)]\mathbf{Q}_n\right\}\mathbf{F}_n^{M^2-1}\boldsymbol{\sigma} \end{bmatrix},$$

$\mathbf{F}_n = E\left[\mathbf{V}^T(n)\otimes\mathbf{V}^T(n)\right]$, $\mathbf{Q}_n = E\left[\mathbf{v}^T(n)\otimes\mathbf{V}^T(n)\right]$, $\boldsymbol{g}_n = \text{vec}\left\{E\left[\mathbf{v}(n)\mathbf{v}^T(n)\right]\right\}$, $\boldsymbol{\sigma} = \text{vec}\{\mathbf{\Sigma}\}$ *and the* p_i *are the coefficients of the characteristic polynomial of* $\mathbf{F}_n$*, i. e.,* $p(x) = x^{M^2} + p_{M^2-1}x^{M^2-1} + \ldots + p_1x + p_0 = \det(x\mathbf{I} - \mathbf{F}_n)$.

Since the matrices $\mathbf{V}(n)$ and $\mathbf{v}(n)$ depend on $\mathbf{w}(n)$, the matrices $E\left[\mathbf{V}(n)\right]$, $E\left[\mathbf{v}(n)\right]$, $\boldsymbol{\mathcal{F}}_n$, $\mathbf{Q}_n$, and $\mathbf{g}_n$ are also function of $E\left[\mathbf{w}(n)\right]$ and of the state vector $\boldsymbol{\mathcal{W}}(n)$. Therefore, the transient behavior of the Fx-SM-AP algorithm is described by the cascade of two nonlinear time invariant systems. A similar characterization was already obtained in [6] for other adaptive algorithms with error nonlinearities. The stability and the steady-state analysis can now be characterized by studying the properties of the cascade of the nonlinear systems.

With the steady-state analysis, we are here interested in evaluating the mean-square-error (MSE) in steady-state, which is defined by $\text{MSE} = \lim_{n\to+\infty} E\left[\sum_{k=1}^{K} e_k^2(n)\right]$. In the hypothesis that $\mathbf{w}(n)$ is uncorrelated with $\sum_{k=1}^{K}\mathbf{u}_k(n)\mathbf{u}_k^T(n)$ and with $\sum_{k=1}^{K} d_k(n)\mathbf{u}_k(n)$, the MSE can be expressed as

$$\text{MSE} = S_d + 2\,\mathbf{R}_{ud}^T\mathbf{w}_\infty + \lim_{n\to+\infty} E\left[\mathbf{w}^T(n)\mathbf{R}_{uu}\mathbf{w}(n)\right], \quad (7)$$

where $S_d = E\left[\sum_{k=1}^{K} d_k^2(n)\right]$, $\mathbf{R}_{uu} = E\left[\sum_{k=1}^{K}\mathbf{u}_k(n)\mathbf{u}_k^T(n)\right]$ and $\mathbf{R}_{ud} = E\left[\sum_{k=1}^{K}\mathbf{u}_k(n)d_k(n)\right]$. The computation of (7) requires the evaluation of $\lim_{n\to+\infty} E\left[\|\mathbf{w}(n)\|_{\mathbf{\Sigma}}\right]$, with $\mathbf{\Sigma} = \mathbf{R}_{uu}$. This limit can be estimated with a methodology similar to [4] and thus the following expression is obtained

$$\text{MSE} = S_d + 2\,\mathbf{R}_{ud}^T\mathbf{w}_\infty + \left(\mathbf{g}_\infty^T - 2\,\mathbf{w}_\infty^T\mathbf{Q}_\infty\right)\left(\mathbf{I} - \mathbf{F}_\infty\right)^{-1}\text{vec}\{\mathbf{R}_{uu}\}, \quad (8)$$

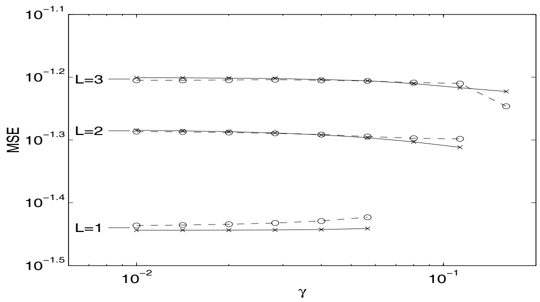
with $\mathbf{F}_\infty = \lim_{n\to+\infty}\mathbf{F}_n$ and similar definitions for $\mathbf{Q}_\infty$ and $\mathbf{g}_\infty$. For small values of the step-size, the matrices $\mathbf{F}_\infty$, $\mathbf{Q}_\infty$, and $\mathbf{g}_\infty$ can be estimated from $\mathbf{F}_n$, $\mathbf{Q}_n$, and $\mathbf{g}_n$, respectively, by considering $\mathbf{w}(n) = \mathbf{w}_\infty$.

An expression similar to that of (8) can also be obtained for the mean-square-deviation in steady-state [5].

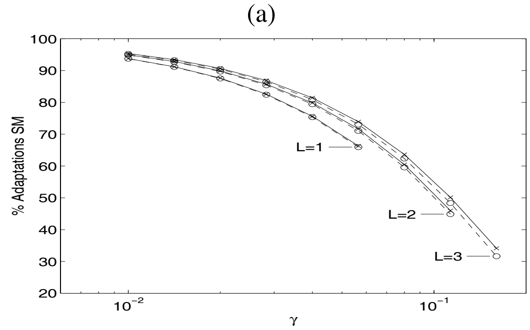
5. EXPERIMENTAL RESULTS

In this section, we show some experimental results obtained with a multichannel active noise control system with $I = 1$, $J = 2$, $K = 2$. The impulse responses of the primary and secondary paths are respectively:
$\mathbf{p}_{11}(n) = [0, 1.0, -0.3, 0.2]$, $\mathbf{p}_{21}(n) = [0, 1.0, -0.2, 0.1]$, $\mathbf{s}_{11}(n) = [0, 1.0, 1.5, -1.0]$, $\mathbf{s}_{12}(n) = [0, 1.0, 1.3, -1.0]$, $\mathbf{s}_{21}(n) = [0, 1.0, 1.3, -1.0]$, $\mathbf{s}_{22}(n) = [0, 1.0, 1.2, -1.0]$. The input signal is a zero-mean, unit-variance colored Gaussian noise with $E[x(n)x(n-m)] = 0.9^{|m|}$ and a zero-mean, white Gaussian noise is added to $d_k(n)$ to get a 30 dB signal-to-noise ratio. Thus, the mean absolute values of $d_1(n)$ and $d_2(n)$ are around 0.71. The controller is a two-channel linear filter with memory length 4, i.e., with $M = 8$.

The evaluation of the asymptotic solution $\mathbf{w}_\infty$, i.e. the estimation of the fixed-point of (6), is performed with an iterative procedure. We first evaluate the asymptotic solution for $\gamma = 0$, i. e., for $f_k(n) = 1$, using (6). We then use this



(a)



(b)

Fig. 2. Estimated values (–) and simulation values (- -) of (a) steady-state MSE and (b) percent of adaptations versus γ for AP orders $L = 1, 2$, and 3.

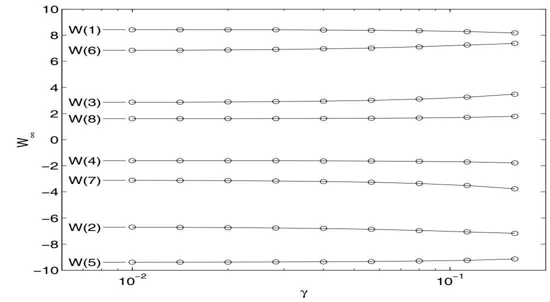


Fig. 3. Coefficients of $\mathbf{w}_\infty$ versus γ for AP order $L = 3$ and $\mu = 0.125$.

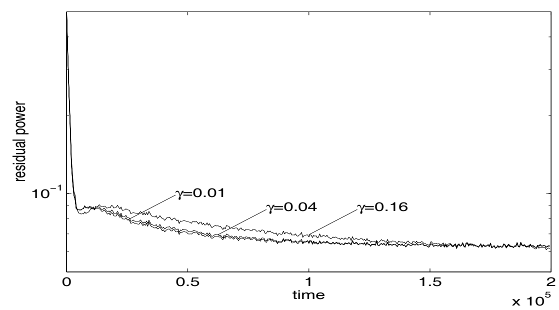


Fig. 4. Evolution of residual error for AP order $L = 3$ and $\mu = 0.125$.

value of $\mathbf{w}_\infty$ for evaluating with better precision $f_k(n)$ and the matrices on the right side of (6). We eventually improve the estimation of $\mathbf{w}_\infty$ by iterating its computation and that of the matrices on the right side of (6). For values of γ reasonably lower than the mean absolute value of $d_k(n)$, the iterative process converges in few steps. On the contrary, for larger values of γ, failure of convergence is observed. In this situation, other more effective procedures should be adopted for estimating the fixed point of (6).

Once $\mathbf{w}_\infty$ is determined, we can evaluate the MSE from (8). Fig. 2-(a) diagrams the MSE of the Fx-SM-AP algorithm, estimated with (8) or obtained from simulations, at different values of γ in the range $[0.01, 0.16]$ and for the AP order $L = 1, 2$, and 3 when the step-size $\mu = 0.125$. Fig. 2-(b) diagrams for the same values of γ the percent of times when $|e_k(n)| > \gamma$, i.e., the percent of adaptations performed at steady-state by using the SM criterion estimated with $\mathbf{w}_\infty$ or obtained from simulation. The missing points in the two diagrams refer to situations where the iterative procedure was unable to converge.

Fig. 3 diagrams the coefficients of $\mathbf{w}_\infty$ for γ in the range $[0.01, 0.16]$, a step-size $\mu = 0.125$, and an AP order $L = 3$. For the same step-size and AP order, Fig. 4 plots the learning curves of the residual error for $\gamma = 0.1, 0.4$, and 0.16.

It is evident from Fig. 2, 3, and 4 that the reduction of the computational complexity obtained by applying the SM criterion affects only slightly the asymptotic solution, the MSE and the convergence speed of the algorithm.

6. CONCLUSIONS

In this paper we have provided an analysis of transient and steady-state behavior of a multichannel Fx-SM-AP algorithm. The analysis relies on energy conservation arguments and it does not apply IT nor it imposes any restriction to the signal distributions. The analysis results show that for an appropriate range of γ the Fx-SM-AP algorithm reduces the computational complexity without trading residual mean-square-error or convergence speed.

7. REFERENCES

[1] G. L. Sicuranza and A. Carini, "Nonlinear multichannel active noise control using partial updates," in *Proc. ICASSP 2005*, Philadelphia, PA, Mar. 2005, pp. iii/109–iii/112.

[2] E. Bjarnason, "Analysis of the filtered-X LMS algorithm," *IEEE Trans. Speech Audio Processing*, vol. 3, no. 6, pp. 504–514, Nov. 1995.

[3] O. J. Tobias, J. M. Bermudez, and N. J. Bershad, "Mean weight behavior of the filtered-X LMS algorithm," *IEEE Trans. Signal Processing*, vol. 48, no. 4, pp. 1061–1075, Apr. 2000.

[4] H. C. Shin and A. H. Sayed, "Mean-square performance of a family of affine projection algorithms," *IEEE Trans. Signal Processing*, vol. 52, no. 1, pp. 90–102, Jan. 2004.

[5] A. Carini and G. L. Sicuranza, "Transient and steady-state analysis of multichannel filtered-X affine projection algorithms," *IEEE Trans. Signal Processing*, vol. 54, no. 2, pp. 665–678, Feb. 2006.

[6] T. Y. Al-Naffouri and A. H. Sayed, "Transient analysis of adaptive filters with error nonlinearities," *IEEE Trans. Signal Processing*, vol. 51, no. 3, pp. 653–663, Mar. 2003.