FREQUENCY ESTIMATION FROM A PARTICULAR ALMOST PERIODIC FUNCTION WITH APPLICATION TO LASER VIBROMETRY SIGNALS

C. Lévy-Leduc

Ecole Nationale Supérieure des Télécommunications - Département TSI
37/39 Rue Dareau - 75014 Paris, FRANCE - Email : levyledu@tsi.enst.fr

ABSTRACT

We consider the problem of estimating the frequency of an unknown periodic function from observations which consist of a particular almost periodic function and additive white noise. First, we propose a frequency estimator based on a penalized least-squares approach and we explain how to implement it in practice. Then, we compare its performances with the optimal ones forecast by a theoretical study. At last, we apply our processing to some signals arising in laser vibrometry and compare it with a classical technique in such a field.

1. INTRODUCTION

Let us consider the following model:

$$X_j = e^{2i\pi f_{dop}\, j\Delta}\, s^\star(j\Delta) + \varepsilon_{1,j} + i\varepsilon_{2,j},\ 1 \le j \le n, \qquad (1)$$

where $s^\star$ is a complex valued unknown periodic function with unknown frequency $f^\star$, f_{dop} is an unknown positive real parameter and the $\varepsilon_{1,j}$'s, $\varepsilon_{2,j}$'s are independent Gaussian random variables of unknown variance $(\sigma^\star)^2$. We also assume that the $\varepsilon_{1,j}$'s and $\varepsilon_{2,j}$'s are independent. We are interested in estimating the frequency $f^\star$ of $s^\star$.

Such a problem arises in laser vibrometry which is a technique used for identifying, with a laser, a target by analyzing its vibrations without any contact with it. Indeed, the signal to analyze after emission of a laser wave and reflection on a vibrating object having a translation motion, satisfies model (1). f_{dop} is the frequency due to the Doppler effect and $f^\star$ is the vibration frequency of the object of interest if its vibrations are assumed to be periodic. Since the vibration frequency characterizes a vibrating object, the estimation of $f^\star$ should lead to its identification.

A relatively close problem has already been solved by several authors in a parametric framework. Indeed, the regression function involved in model (1) is *almost periodic* and thus can be approximated by $\sum_{k=1}^{K} a_k \exp(i\lambda_k t)$ for some $\lambda_k \in \mathbb{R}$. Many papers study the parametric case which consists in estimating $(\lambda_k,\ 1 \le k \le K)$ from observations satisfying the previous model when the regression function is $\sum_{k=1}^{K} a_k \exp(i\lambda_k t)$, K being known or not, see [1, 2].

Thanks to the company Thales Optronique for funding.

When K is large, a natural solution is to use a semiparametric approach. In the previous model, when $\lambda_k = k\lambda_1$, it consists in considering the regression function as an unknown periodic function and just estimating the parameter λ_1. This idea has already been used by [3] who estimate $f^\star$ from model (1) in the particular case when the regression function is periodic, *i.e* when f_{dop} is equal to zero or to a multiple of $f^\star$. [3] propose in this case a consistent and efficient estimator of $f^\star$ and [4] have implemented a method for estimating the frequency of a periodic function in additive Gaussian white noise without making any assumption on its shape.

In this paper, we exhibit in the semiparametric framework of model (1), an estimator of $f^\star$ which we implement and apply to some synthetic data arising in laser vibrometry, thus improving existing methods in this field.

2. THE ESTIMATION METHOD

Estimating $f^\star$ from model (1) using the maximum likelihood approach consists in minimizing the criterion

$$\sum_{j=1}^{n} \left| X_j - \sum_{k=-K}^{K} a_k\, e^{2i\pi(kf+f_d)j\Delta} \right|^2 \qquad (2)$$

with respect to f_d, f and the coefficients $(a_k,\ -K \le k \le K)$ if the unknown periodic function $s^\star$ is approximated by a trigonometric polynomial: $s(t) = \sum_{k=-K}^{K} a_k\, e^{2ik\pi f t}$.

We can easily prove that the minimization of (2) with respect to (a_k) is approximated, for large n, by

$$\sum_{j=1}^{n} \left| X_j - \sum_{k=-K}^{K} d_k(f, f_d)\, e^{2i\pi(kf+f_d)j\Delta} \right|^2 \qquad (3)$$

where $d_k(f, f_d) = n^{-1} \sum_{j=1}^{n} X_j\ \exp(-2i\pi(kf + f_d)j\Delta)$. The quantity (3) can be approximated, for large n, by

$$\sum_{j=1}^{n} |X_j|^2 - n \sum_{k=-K}^{K} |d_k(f, f_d)|^2 .$$

Let $\Lambda_K(f, f_d) = \sum\limits_{k=-K}^{K} |d_k(f, f_d)|^2$. Thus, for every $K \ge 1$, an approximated least-squares estimator of $f^\star$ is

$$\hat{f}(K) = \operatorname{Arg}\sup_{f} \left(\sup_{f_d} \Lambda_K(f, f_d) \right). \quad (4)$$

Without any information on the real number of harmonics of $s^\star$, K has also to be estimated. Indeed, one can see that maximizing $\sup_{f_d} \Lambda_K(f, f_d)$ with respect to both K and f may lead to overestimate the real number of harmonics of $s^\star$ and to underestimate $f^\star$.

To avoid this overestimation of K, we shall use a penalization approach where $f^\star$ is estimated by $\hat{f}(\hat{K}(\beta))$, β being a positive number to be chosen conveniently (we shall explain how to do this in section 3) and

$$\hat{K}(\beta) = \operatorname*{Arg\,inf}_{1 \le K \le K_{max}} \left(\frac{1}{n} \sum_{j=1}^{n} |X_j|^2 - \sup_{(f, f_d)} \Lambda_K(f, f_d) + \beta K \right).$$

Note that K_{max} is an upper bound for the number of the most significant positive harmonics of the periodic signal $s^\star$. Thus defined, the estimator of $f^\star$ is based on a penalized least-squares criterion since $1/n \sum_{j=1}^{n} |X_j|^2 - \sup_{(f,f_d)} \Lambda_K(f, f_d)$ can be seen as an approximation of the residual variance when a shifted trigonometric polynomial of degree K is fitted to the observed sequence (X_j).

3. PRACTICAL IMPLEMENTATION ILLUSTRATED ON PARTICULAR SYNTHETIC DATA

In this section, we shall explain how to compute the previous estimator from synthetic data arising in laser vibrometry. If we aim at analyzing the vibrations of an object consisting of M punctual reflectors vibrating in a sinusoidal way, $s^\star$ can be written as follows

$$s^\star(t) = \sum_{m=1}^{M} a_m \exp\left(\frac{4i\pi\alpha_m}{\lambda} \cos(2\pi f^\star t + \varphi) \right), \quad (5)$$

where a_m is the amplitude of the signal reflected by the reflector number m, α_m its vibration amplitude, λ the laser wavelength, $f^\star$ is the vibration frequency of the vibrating object and $\varphi \in \mathbb{R}$. That is why we shall explain how to estimate the frequency of the following simulated complex valued signal:

$$s^\star(t) = a \exp(ic \cos(2\pi f^\star t)),$$

where $a = 0.08$, $c = 4\pi \times 20 \times 10^{-6}/\lambda$ with $\lambda = 1.5 \times 10^{-6}$ and $f^\star = 30$ Hz from the data: $X_j = e^{2i\pi j f_{dop}/n} s^\star(j/n) + \varepsilon_{1,j} + i\varepsilon_{2,j},\ 1 \le j \le n$, with $f_{dop} = 200$ Hz. Thus defined, $s^\star$ is a periodic function of frequency 30 Hz with about 168 (positive) harmonics at 30, 60, 90,..., 5040 Hz.

Figure 1-a displays the imaginary part of $(s^\star(j/n),\ j = 1, \cdots, 15000)$ with $n = 2^{18}$. Figure 1-b displays the periodogram of $(s^\star(j/n),\ j = 1, \cdots, n)$ *i.e* the squared modulus

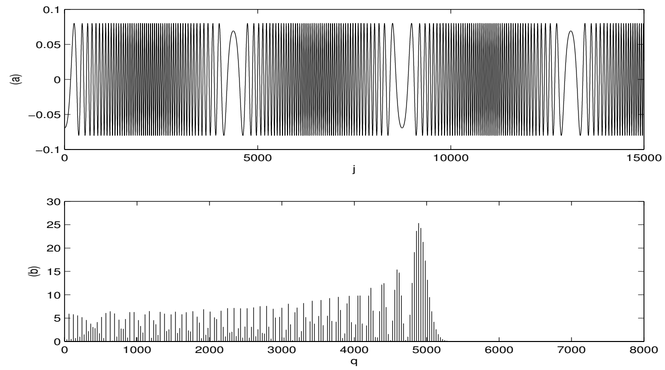

Fig. 1. A synthetic signal ; (a) the imaginary part of $(s^\star(j/n),\ 1 \le j \le 15000)$, (b) the periodogram of $s^\star$.

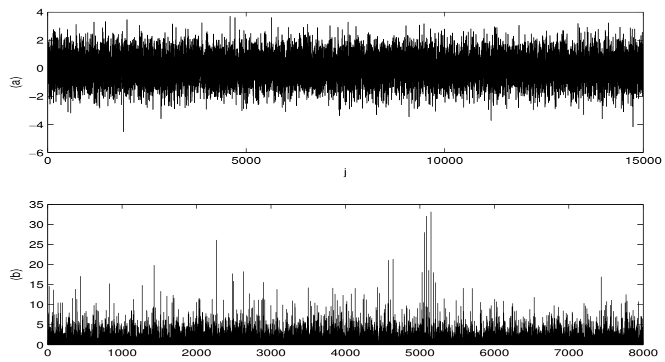

Fig. 2. The observed signal X ; (a) the imaginary part of $X = (X_j,\ 1 \le j \le 15000)$, (b) the periodogram of X.

of the discrete Fourier transform (DFT) of $s^\star$ and computed by a fast Fourier transform (FFT):

$$I_{s^\star}(q) = \frac{1}{n} \left| \sum_{j=1}^{n} e^{-\frac{2i\pi qj}{n}} s^\star\left(\frac{j}{n}\right) \right|^2,\ 1 \le q \le 8000.$$

The periodogram of $\left(e^{2i\pi f_{dop} j/n} s^\star(j/n)\right)$ is the same as the one of $s^\star$ except that it is shifted. The observed sequence $(X_j,\ j = 1, \cdots, n)$ satisfies:

$$X_j = e^{2i\pi f_{dop} j/n} s^\star(j/n) + \varepsilon_{1,j} + i\varepsilon_{2,j},\ 1 \le j \le n, \quad (6)$$

where $(\varepsilon_{1,j})$ and $(\varepsilon_{2,j})$ are independent Gaussian random variables with zero-mean and unit variance. Figure 2 displays the imaginary part of $(X_j,\ j = 1, \cdots, 15000)$ and its periodogram. The signal to noise ratio is so low that the deterministic part of the observations cannot be visually detected.

3.1. Maximization of Λ_K

For any $K \ge 1$, it is impossible to maximize $\Lambda_K(f, f_d)$ with respect to (f, f_d) in a closed-form. A numerical solution is to maximize Λ_K on a grid: for any $(f, f_d) \in \mathbb{R}_+^2$ and for any $K \geq 1$, we define

$$\tilde{\Lambda}_K(f, f_d) = \frac{1}{n^2} \sum_{k=-K}^{K} |\hat{X}(]kf + f_d[)|^2,$$

where $]u[$ is the nearest integer to the real number u, $\hat{X}$ is the classical discrete Fourier transform of X defined by

$$\hat{X}(l) = \sum_{j=1}^{n} e^{-\frac{2il\pi j}{n}} X_j, \; 0 \leq l \leq n-1.$$

Let $(f_m, \; m = 1, \cdots, M)$ be the elements of a regular grid with $f_{m+1} - f_m \leq 1/K$. This ensures that $]kf[=]kf_m[$, for any $f \in [f_m, f_{m+1})$ and any $1 \leq k \leq K$. Finally, we shall take in the following simulations the f_m's in the interval $[10, 100]$ Hz with $f_{m+1} - f_m = 0.001$ and f_d in a grid $(f_{d,p}, \; p = 1, \ldots, P)$ on $[10, 100]$ Hz such that $f_{d,p+1} - f_{d,p} = 0.01$. Note that the $f_{d,p}$'s are in the same interval than the f_m's since we can assume that $f_{dop} < f^\star$. Indeed, otherwise, $f_{dop} = lf^\star + f_r$, where $l \geq 1$, $f_r < f^\star$ and $\exp(2i\pi f_{dop} t) \; s^\star(t) = \exp(2i\pi f_r t) \; \tilde{s}(t)$ where $\tilde{s}$ is a periodic signal having the same period as $s^\star$.

Thus, for any $K \geq 1$, we can compute an estimate of $f^\star$ defined by

$$\hat{f}(K) = \operatorname*{Arg\,sup}_{f \in \{f_m\}} \left(\sup_{f_d \in \{f_{d,p}\}} \tilde{\Lambda}_K(f, f_d) \right).$$

3.2. Choice of K

Let $J(K) = \frac{1}{n} \sum_{j=1}^{n} |X_j|^2 - \sup_{f_d \in \{f_{d,p}\}} \tilde{\Lambda}_K(\hat{f}(K), f_d)$.
By the following Lemma, we have a way to compute easily a convenient β and consequently $\hat{K}(\beta)$ which has already been used in [4].

Lemma 1. *There exist two sequences $K_1 = 1 < K_2 < \cdots$, and $\beta_0 = +\infty > \beta_1 > \cdots$, with*

$$\beta_p = \max_{K_p < K \leq K_{max}} \frac{J(K) - J(K_p)}{K_p - K} = \frac{J(K_{p+1}) - J(K_p)}{K_p - K_{p+1}}, p \geq 1$$

and such that $\forall \beta \in (\beta_p; \beta_{p-1}]$, $\hat{K}(\beta) = K_p$.

We propose to choose the number $\hat{K}(\beta)$ that maximizes the "second derivative" of the sequence $(K_p, J(K_p))$, *i.e* $K_{\hat{p}}$, where $\hat{p}$ maximizes $l_p = \beta_{p-1} - \beta_p$.

Let us explain this choice with our synthetic example. Figure 3-a displays the sequence $(K, J(K))$ and Figure 3-b displays the associated sequence $(K_p, J(K_p))$ obtained from the synthetic signal (X_j). We clearly see on this Figure that the residual variance decreases much more for $1 \leq K \leq 163$ than for $K > 163$. In other words, a good trade-off between the fit with the data and the number of harmonics is obtained with $K = 163$, that is, for the value of K that maximizes the second derivative of the sequence $(K_p, J(K_p))$.

Finally, the estimation algorithm can be summarized as follows:

1. for $K = 1, \ldots, K_{Max}$, compute $J(K)$,
2. compute the sequences (K_p) and (β_p) and define $l_p = \beta_{p-1} - \beta_p$
3. let $\hat{p} = \operatorname*{Arg\,sup}_{p \geq 1} \{l_p\}$. Then, set $\hat{f} = \hat{f}(\hat{K}_{\hat{p}})$.

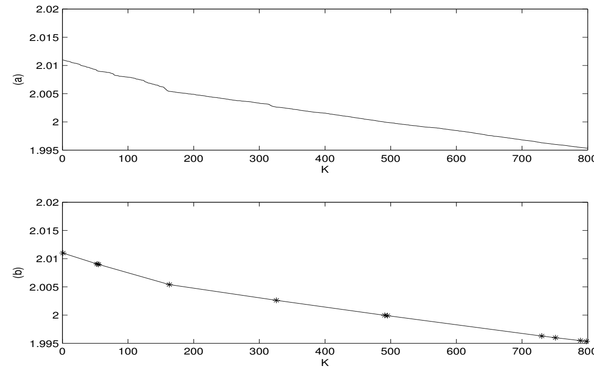

Fig. 3. The sequence $(K, J(K))$, (b) the sequence $(K_p, J(K_p))$: '*'.

Table 1 displays the numbers (K_p), the estimated frequencies $(\hat{f}(K_p))$ and the lengths (l_p) of the intervals $((\beta_p; \beta_{p-1}])$. Here, the maximization of l_p yields $\hat{K}_{\hat{p}} = 163$ and $\hat{f} = 30$Hz which is the true value of $f^\star$.

$\hat{K}_\beta$	$\hat{f}(\hat{K}(\beta))$	$(\beta_p, \beta_{p-1}] \times 10^5$	$(\beta_{p-1} - \beta_p) \times 10^6$
53	89.98	(3.53, 3.67]	1.39
55	89.98	(3.32, 3.53]	2.13
163	30	(1.71, 3.32]	16.1
326	15	(1.61, 1.71]	0.92
491	15	(1.59, 1.61]	0.23
495	15	(1.53, 1.59]	0.58
730	15	(1.44, 1.53]	0.97
751	15	(1.37, 1.44]	0.63
789	15	(1.31, 1.37]	0.58
798	15	(1.01, 1.31]	3.08

Table 1. The values of $\hat{K}(\beta)$ and $\hat{f}(\hat{K}(\beta))$ as functions of β

4. COMPARISON WITH THEORETICAL PROPERTIES

A theoretical study driven in [5] proves that an estimator $\tilde{f}_n$ of $f^\star$ in model (1) is asymptotically efficient if it satisfies

$$n^{3/2} \Delta(\tilde{f}_n - f^\star) \rightarrow \mathcal{N}\left(0, \frac{J_2}{J_1 J_2 - J_3^2}\right) \text{ in distribution,}$$

where $J_1 = \frac{1}{12(\sigma^\star)^2} \int_0^1 |s_0'|^2(t) dt$, $J_2 = \frac{\pi^2}{3(\sigma^\star)^2} \int_0^1 |s_0|^2(t) dt$, $J_3 = \frac{\pi}{6(\sigma^\star)^2} i \int_0^1 \overline{s_0'}(t) s_0(t) dt$, the function s_0 satisfying:

$s^\star(t) = s_0(f^\star t)$.

To compare the performances of our practical procedure with the optimal ones, we have simulated $L = 50$ observed series (X_j) satisfying (6) with

$$s^\star(t) = \sum_{k=1}^{3} a \exp(2ik\pi f^\star t) + b \exp(-2ik\pi f^\star t),$$

for $f^\star = 3.212$ Hz, $f_{dop} = 10$ Hz, $\sigma^\star = 1$ and for each value of (a, b) in Table 2. Then, for each of these values of (a, b), we have computed the root mean squared error (RMSE) defined by: $\sqrt{L^{-1}\sum_{l=1}^{L}(\hat{f}_l - f^\star)^2}$ where $(\hat{f}_l,\ l = 1, \ldots, L)$ are the L estimations of $f^\star$ obtained with our algorithm. The results are gathered in Table 2. We can remark that the values of the RMSE's are close to those forecast by the theoretical approach, for n large enough.

a	0.25	0.25	0.25
b	0.15	0.2	0.25
$\frac{1}{n^{3/2}\Delta}\sqrt{\frac{J_2}{J_1 J_2 - J_3^2}}$ $(n = 2^{10})$	0.0257	0.0221	0.0195
RMSE $(n = 2^{10})$	0.0541	0.0762	0.0569
$\frac{1}{n^{3/2}\Delta}\sqrt{\frac{J_2}{J_1 J_2 - J_3^2}}$ $(n = 2^{11})$	0.0182	0.0156	0.0137
RMSE $(n = 2^{11})$	0.0214	0.0209	0.0465
$\frac{1}{n^{3/2}\Delta}\sqrt{\frac{J_2}{J_1 J_2 - J_3^2}}$ $(n = 2^{12})$	0.0129	0.0119	0.0097
RMSE $(n = 2^{12})$	0.0169	0.0142	0.0147

Table 2. Comparison of the best asymptotic standard deviation given by the theory with the RMSE

5. COMPARISON WITH THE MICRODOPPLER

In this section, we aim to compare the performances of our algorithm with those of the *microdoppler* developed in [6] and classically used to analyze the vibrations of an object having a translation motion with a laser.

In the following, we shall more particularly be interested in "long-range" vibrations analysis which means that the object is far from the laser that is why we have to provide a processing that can accurately estimate $f^\star$ at signal to noise ratio as low as possible. Indeed, the further away the vibrating object is, the lower the signal to noise ratio (SNR) is, the SNR being defined for observations (X_j) satisfying (1) with $s^\star$ of the form (5) by

$$SNR = 10\log_{10}\left(\frac{\sum_{m=1}^{M} a_m^2}{2\sigma^{\star 2}}\right), \text{ (in dB)}.$$

We shall see in this section that our method gives an accurate estimation of $f^\star$ at smaller SNR than those required by the microdoppler approach. For each signal $s^\star$, each value of f_{dop} and each value of SNR, we have simulated $L = 20$ observed series satisfying (6) and estimated $f^\star$ thanks to the two methods proposed previously. The quality of an estimator is quantified by computing its RMSE: R_m for the microdoppler and R_p for our method.

	SNR	-30	-25	-20	-15	-10	-5
ex. 1	R_m	31.69	34	45.44	26.78	0.2	0.2
	R_p	27.37	7.2	0	0	0	0
ex. 2	R_m	42.79	49.36	37.11	40.85	26.16	0.4
	R_p	45.1	18.12	0	0	0	0

Table 3. Root mean squared errors for the microdoppler (R_m) and our method (R_p), obtained with different signals and different SNR.

1. In this example, $s^\star(t) = a\sum_{l=1}^{20}\exp(ic_l\cos(2\pi f^\star t))$ where $f^\star = 32.2$ Hz, $f_{dop} = 225$ Hz, $n = 2^{18}$ and $c_l = 4\pi\alpha_l/\lambda$ with $\lambda = 1.5\times 10^{-6}$ and $\alpha_l = 55 + l,\ l = 1, \cdots, 20$.
2. The parameters remain unchanged except that $f^\star = 83.4$ Hz, $f_{dop} = 200$ Hz and $\alpha_l = 10 + 0.5l,\ l = 1, \cdots, 20$.

All these results let us believe that our method is promising in the laser vibrometry field.

6. REFERENCES

[1] B. G. Quinn and E. J. Hannan, *The estimation and tracking of frequency*, Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2001.

[2] P. J. Kootsookos, "A review of the frequency estimation and tracking problems," Frequency estimation and tracking project, CRC for Robust and Adaptive Systems, DSTO, Salisbury Site, February 1999, www.itee.uq.edu.au/ kootsoop/comparaison-t.pdf.

[3] E. Gassiat and C. Lévy-Leduc, "Efficient semiparametric estimation of the periods in a superposition of periodic functions with unknown shape," submitted, 2005, http://www.math.u-psud.fr/~levyleduc/gassiatlevy.ps.

[4] M. Lavielle and C Lévy-Leduc, "Semiparametric estimation of the frequency of unknown periodic functions and its application to laser vibrometry signals," *IEEE Transactions on Signal Processing*, vol. 53, no. 7, pp. 2306–2314, 2005.

[5] C. Lévy-Leduc, *Estimation semi-paramétrique de la fréquence de fonctions périodiques dans divers modèles statistiques : théorie et pratique*, Ph.D. thesis, Université Paris-Sud, 2004, http://www.math.u-psud.fr/~levyleduc/Theselevyleduc.pdf.

[6] I. Renhorn, C. Karlsson, D. Letalick, M. Millnert, and R. Rutgers, "Coherent laser radar for vibrometry: robust design and adaptive signal processing," *SPIE*, vol. 2472, pp. 23–30, 1995.