MODEL ORDER SELECTION RULE FOR ESTIMATING THE PARAMETERS OF 2-D SINUSOIDS IN COLORED NOISE

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ABSTRACT

We consider the problem of jointly estimating the number as well as the parameters of two-dimensional sinusoidal signals, observed in the presence of an additive colored noise field. In this framework we consider the problem of least squares estimation of the parameters of 2-D sinusoidal signals observed in the presence of an additive noise field, when the assumed number of sinusoids is incorrect. In the case where the number of sinusoidal signals is under-estimated we show the almost sure convergence of the least squares estimates to the parameters of the dominant sinusoids. In the case where the number of sinusoidal signals is over-estimated, the estimated parameter vector obtained by the least squares estimator contains a sub-vector that converges almost surely to the correct parameters of the sinusoids. Based on these results, we prove the strong consistency of a large family of model order selection rules.

1. INTRODUCTION

We consider the problem of jointly estimating the number as well as the parameters of two-dimensional sinusoidal signals, observed in the presence of an additive noise field. This problem is, in fact, a special case of a much more general problem: From the 2-D Wold-like decomposition we have that any 2-D regular and homogeneous discrete random field can be represented as a sum of two mutually orthogonal components: a purely-indeterministic field and a deterministic one. In this paper we consider the special case where the deterministic component consists of a finite (unknown) number of sinusoidal components, while the purely-indeterministic component is assumed to be a colored infinite order non-symmetric half plane, or quarter-plane, moving average noise field.

Many algorithms have been devised to estimate the parameters of sinusoids observed in additive colored noise. Most of these assume the number of sinusoids is *a-priori* known. However this assumption does not always hold in practice. In the past three decades the problem of model order selection for 1-D signals has received considerable attention. In general, model order selection rules are based (directly or indirectly) on three popular criteria: Akaike information criterion (AIC), the minimum description length (MDL), and the maximum a-posteriori probability criterion (MAP). All these criteria have a common form composed of two terms: a data term and a penalty term, where the data term is the log-likelihood function evaluated for the assumed model.

Most of the papers that address the problem of model order selection are concerned with various models of one-dimensional signals, while the problem of modelling multidimensional fields has received considerably less attention. In [2], a maximum *a-posteriori* (MAP) model order selection criterion for jointly estimating the number and the parameters of two-dimensional sinusoids observed in the presence of an additive white Gaussian noise field, is derived. In [3], we proved the strong consistency of a large family of model order selection rules, which includes the MAP based rule in [2] as a special case.

In this paper we extend the results of [3] to the case were the additive noise is colored and have infinite order non-symmetric half-plane or quarter-plane moving average representation.

2. NOTATIONS, DEFINITIONS AND ASSUMPTIONS

Let $\{y(n,m)\}$ be a real valued field,

$$y(n,m) = \sum_{i=1}^{P} \rho_i^0 \cos(\omega_i^0 n + \upsilon_i^0 m + \varphi_i^0) + w(n,m), \quad (1)$$

where $0 \leq n \leq N-1$, $0 \leq m \leq M-1$ and for each i, ρ_i^0 is non-zero. Due to physical considerations it is further assumed that for each i, $|\rho_i^0|$ is bounded .

Recall that the *non-symmetrical half-plan total-order* is defined by

$$(i,j) \succeq (s,t) \text{ iff}$$
$$(i,j) \in \{(k,l) | k = s, l \geq t\} \cup \{(k,l) | k > s, -\infty \leq l \leq \infty\} \quad (2)$$

We make the following assumptions:

Assumption 1: The stationary noise field $\{w(n,m)\}$ can be represented by an infinite order non-symmetric half-plane MA noise field, i.e.,

$$w(n,m) = \sum_{(r,s) \succeq (0,0)} a(r,s) u(n-r, m-s) \quad (3)$$

where the field $\{u(n,m)\}$ is an i.i.d. real valued zero-mean random field with finite second order moment, σ^2, and satisfy the condition $E[|u(0,0)|^\alpha] < \infty$ for some $\alpha > 3$ .The sequence $a(i,j)$ is an absolutely summable deterministic sequence with

$$\sum_{(r,s) \succeq (0,0)} |a(r,s)| < \infty. \quad (4)$$

Let $f_w(\omega, \upsilon)$ denote the spectral density function of the noise field $\{w(n,m)\}$. Hence,

$$f_w(\omega,\upsilon) = \sigma^2 \left| \sum_{(r,s) \succeq (0,0)} a(r,s) e^{j(\omega r + \upsilon s)} \right|^2 \quad (5)$$

Assumption 2: The spatial frequencies $(\omega_i^0, \upsilon_i^0) \in (0, 2\pi) \times (0, 2\pi)$, $1 \leq i \leq P$ are pairwise different. In other words, $\omega_i^0 \neq \omega_j^0$ or $\upsilon_i^0 \neq \upsilon_j^0$, when $i \neq j$.

Let $\{\Psi_i\}$ be a sequence of rectangles such that $\Psi_i = \{(n, m) \in \mathcal{Z}^2 \mid 0 \leq n \leq N_i - 1, 0 \leq m \leq M_i - 1\}$.

Definition 1: The sequence of subsets $\{\Psi_i\}$ is said to tend to infinity (we adopt the notation $\Psi_i \to \infty$) as $i \to \infty$ if

$$\lim_{i\to\infty} \min(N_i, M_i) = \infty$$

and

$$0 < \lim_{i\to\infty} (N_i/M_i) < \infty.$$

To simplify notations, we shall omit in the following the subscript i. Thus, the notation $\Psi(N, M) \to \infty$ implies that both N and M tend to infinity as functions of i, and at roughly the same rate.

Definition 2: Let Θ_k be a bounded and closed subset of the $4k$ dimensional space $\mathrm{R}^k \times ((0, 2\pi) \times (0, 2\pi))^k \times [0, 2\pi)^k$ where for any vector $\theta_k = (\rho_1, \omega_1, \upsilon_1, \varphi_1, \ldots, \rho_k, \omega_k, \upsilon_k, \varphi_k) \in \Theta_k$ the coordinate ρ_i is non-zero and bounded for every $1 \leq i \leq k$ while the pairs (ω_i, υ_i) are pairwise different, so that no two regressors coincide. We shall refer to Θ_k as the *parameter space*.

From the model definition (1) and the above assumptions it is clear that $\theta_k^0 = (\rho_1^0, \omega_1^0, \upsilon_1^0, \varphi_1^0, \ldots, \rho_k^0, \omega_k^0, \upsilon_k^0, \varphi_k^0) \in \Theta_k$.

Define the loss function due to the error of the k-th order regression model

$$\mathcal{L}_k(\theta_k) = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left(y(n,m) - \sum_{i=1}^{k} \rho_i^0 \cos(\omega_i^0 n + \upsilon_i^0 m + \varphi_i^0) \right)^2 . \tag{6}$$

A vector $\hat{\theta_k} \in \Theta_k$ that minimizes $\mathcal{L}_k(\theta_k)$ is called the *Least Square Estimate* (LSE). In the case where $k = P$, the LSE is a *strongly consistent* estimator of θ_P^0 (see, *e.g.*, [5] and the references therein).

3. STRONG CONSISTENCY OF THE OVER- AND UNDER-DETERMINED LSE

In the following, we establish the strong consistency of the above LSE when the number of sinusoids is under-estimated, or over-estimated. Detailed proofs can be found in [4].

The first theorem establishes the strong consistency of the least squares estimator in the case where the number of the regressors is lower than the actual number of sinusoids. Let k denote the assumed number of observed 2-D sinusoids, where $k < P$. To formulate the main result of this section, we shall need an additional assumption:

Assumption 3: For convenience, and without loss of generality, we assume that the sinusoids are indexed according to a descending order of their amplitudes, *i.e.*,

$$\rho_1^0 \geq \rho_2^0 \geq \ldots \rho_k^0 > \rho_{k+1}^0 \ldots \geq \rho_P^0 > 0 , \tag{7}$$

where we assume that for a given k, $\rho_k^0 > \rho_{k+1}^0$ to avoid trivial ambiguities resulting from the case where the k-th dominant component is not unique.

Theorem 1 *Let Assumptions 1-3 be satisfied. Then, the k-regressors parameter vector*

$$\hat{\theta}_k = (\hat{\rho}_1, \hat{\omega}_1, \hat{\upsilon}_1, \hat{\varphi}_1, \ldots, \hat{\rho}_k, \hat{\omega}_k, \hat{\upsilon}_k, \hat{\varphi}_k)$$

that minimizes (6) is a strongly consistent estimator of

$$\theta_k^0 = (\rho_1^0, \omega_1^0, \upsilon_1^0, \varphi_1^0, \ldots, \rho_k^0, \omega_k^0, \upsilon_k^0, \varphi_k^0)$$

as $\Psi(N, M) \to \infty$. That is,

$$\hat{\theta}_k \to \theta_k^0, \ \ a.s. \ as \ \ \Psi(N, M) \to \infty. \tag{8}$$

Proof: See [4] for a detailed proof. ∎

The second theorem establishes the strong consistency of the least squares estimator in the case where the number of the regressors is higher than the actual number of sinusoids. Let k denote the assumed number of observed 2-D sinusoids, where $k > P$. Without loss of generality, we can assume that $k = P + 1$, (as the proof for $k \geq P + 1$ follows immediately by repeating the same arguments). Let the periodogram (scaled by a factor of 2) of the field $\{w(n, m)\}$ be given by

$$I_w(\omega, \upsilon) = \frac{2}{NM} \left| \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} w(n, m) e^{-j(n\omega + m\upsilon)} \right|^2 . \tag{9}$$

The parameter space Θ_P, Θ_{P+1} are defined as in Definition 2.

Theorem 2 *Let Assumptions 1-2 be satisfied. Then, the parameter vector*

$$\begin{aligned} \hat{\theta}_{P+1} \quad = \quad & (\hat{\rho}_1, \hat{\omega}_1, \hat{\upsilon}_1, \hat{\varphi}_1, \ldots, \\ & \hat{\rho}_P, \hat{\omega}_P, \hat{\upsilon}_P, \hat{\varphi}_P, \hat{\rho}_{P+1}, \hat{\omega}_{P+1}, \hat{\upsilon}_{P+1}, \hat{\varphi}_{P+1}) \in \Theta_{P+1} \end{aligned}$$

that minimizes (6) with $k = P + 1$ regressors as $\Psi(N, M) \to \infty$ is composed of the vector $\hat{\theta}_P = (\hat{\rho}_1, \hat{\omega}_1, \hat{\upsilon}_1, \hat{\varphi}_1, \ldots, \hat{\rho}_P, \hat{\omega}_P, \hat{\upsilon}_P, \hat{\varphi}_P)$ which is a strongly consistent estimator of

$$\theta_P^0 = (\rho_1^0, \omega_1^0, \upsilon_1^0, \varphi_1^0, \ldots, \rho_P^0, \omega_P^0, \upsilon_P^0, \varphi_P^0)$$

as $\Psi(N, M) \to \infty$; of the pair of spatial frequencies $(\hat{\omega}_{P+1}, \hat{\upsilon}_{P+1})$ that maximizes the periodogram of the observed realization of the field $\{w(n, m)\}$, i.e.,

$$(\hat{\omega}_{P+1}, \hat{\upsilon}_{P+1}) = \underset{(\omega,\upsilon)\in(0,2\pi)^2}{\arg\max} \ I_w(\omega, \upsilon) \tag{10}$$

and of the element $\hat{\rho}_{P+1}$ that satisfies

$$\hat{\rho}_{P+1}^2 = \frac{2}{NM} I_w(\hat{\omega}_{P+1}, \hat{\upsilon}_{P+1}) . \tag{11}$$

Proof: See [4] for a detailed proof. ∎

In the above theorems, we have considered the problem of least squares estimation of the parameters of 2-D sinusoidal signals observed in the presence of an additive colored noise field, when the assumed number of sinusoids is incorrect. In the case where the number of sinusoidal signals is under-estimated we have established the almost sure convergence of the least squares estimates to the parameters of the dominant sinusoids. This result can be intuitively explained using the basic principles of least squares estimation: Since the least squares estimate is the set of model parameters that minimizes the ℓ_2 norm of the error between the observations and the assumed model, it follows that in the case where the model order is under-estimated the minimum error norm is achieved when the k most dominant sinusoids are correctly estimated. Similarly, in the case where the number of sinusoidal signals is over-estimated, the estimated parameter vector obtained by the least squares estimator contains a $4P$-dimensional sub-vector that converges almost surely to the correct parameters of the sinusoids, while the remaining $k-P$ components assumed to exist, are assigned to the $k - P$ most dominant spectral peaks of the noise power to further minimize the norm of the estimation error.

4. STRONG CONSISTENCY OF A FAMILY OF MODEL ORDER SELECTION RULES

In this section we employ the results derived in the previous section in order to establish the strong consistency of a large family of model order selection rules.

It is assumed that there are Q competing models, where $Q > P$ and finite., and that each model is equiprobable. Following the MDL-MAP template, define the statistic

$$\chi_\xi(k) = NM \log \mathcal{L}_k(\hat{\theta}_k) + \xi k \log NM, \tag{12}$$

where ξ is some finite constant to be specified later,
$k \in \{0, 1, 2, \ldots, Q-1\}$ and $\mathcal{L}_k(\hat{\theta}_k)$ is the minimal value of the error variance of the least square estimator.

The number of 2-D sinusoids is estimated by minimizing $\chi_\xi(k)$ over $k \in \{0, 1, 2, \ldots, Q-1\}$, *i.e.*,

$$\hat{P} = \arg\min_{k \in Z_Q} \left\{ \chi_\xi(k) \right\} \tag{13}$$

Let

$$\mathcal{A} := \frac{\sum_{(r,s)\succeq(0,0)} \sum_{(q,t)\succeq(0,0)} |a(r,s)a(q,t)|}{\sum_{(r,s)\succeq(0,0)} a^2(r,s)} \tag{14}$$

The objective of the next theorem is to prove the asymptotic consistency of the model order selection procedure in (13).

Theorem 3 *Let Assumptions 1-2 be satisfied. Let $\hat{P}$ be given by (13) with $\xi > 14\mathcal{A}$. Then as $\Psi(N,M) \to \infty$*

$$\hat{P} \to P \ \text{a.s.} \tag{15}$$

Proof:
For $k \leq P$,

$$\begin{aligned} &\chi_\xi(k-1) - \chi_\xi(k) \\ &= NM \log \mathcal{L}_{k-1}(\hat{\theta}_{k-1}) + \xi(k-1)\log NM \\ &- NM \log \mathcal{L}_k(\hat{\theta}_k) - \xi k \log NM \\ &= NM \log \left(\frac{\mathcal{L}_{k-1}(\hat{\theta}_{k-1})}{\mathcal{L}_k(\hat{\theta}_k)} \right) - \xi \log NM \end{aligned} \tag{16}$$

From Theorem 1 as $\Psi(N,M) \to \infty$

$$\hat{\theta}_k \to \theta_k^0 \ \text{a.s.} \tag{17}$$

and

$$\hat{\theta}_{k-1} \to \theta_{k-1}^0 \ \text{a.s.} \tag{18}$$

From the definition of $\mathcal{L}_k(\hat{\theta}_k)$, as $\Psi(N,M) \to \infty$

$$\begin{aligned} \mathcal{L}_k(\hat{\theta}_k) &= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left(y(n,m) - \sum_{i=1}^{k} \hat{\rho}_i \cos(\hat{\omega}_i n + \hat{\upsilon}_i m + \hat{\varphi}_i) \right)^2 \\ &= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \Bigg(\sum_{i=1}^{P} \rho_i^0 \cos(\omega_i^0 n + \upsilon_i^0 m + \varphi_i^0) + w(n,m) \\ &- \sum_{i=1}^{k} \hat{\rho}_i \cos(\hat{\omega}_i n + \hat{\upsilon}_i m + \hat{\varphi}_i) \Bigg)^2 \\ &\to \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left(\sum_{i=k+1}^{P} \rho_i^0 \cos(\omega_i^0 n + \upsilon_i^0 m + \varphi_i^0) + w(n,m) \right)^2 . \end{aligned} \tag{19}$$

Recall that for $\omega \in (0, 2\pi)$

$$\sum_{n=0}^{N-1} \cos(\omega n + \varphi) = O(1) . \tag{20}$$

From Lemma 3, [4] we know that as $\Psi(N,M) \to \infty$

$$\sup_{\omega,\upsilon} \left| \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} w(n,m) \cos(\omega n + \upsilon m) \right| \to 0. \ \text{a.s.} \tag{21}$$

Hence, from the SLLN, (20) and (21), we conclude that as $\Psi(N,M) \to \infty$

$$\mathcal{L}_k(\hat{\theta}_k) \to \sigma^2 \sum_{(r,s)\succeq(0,0)} a^2(r,s) + \sum_{i=k+1}^{P} \frac{(\rho_i^0)^2}{2} \ \text{a.s.} \tag{22}$$

and similarly

$$\mathcal{L}_{k-1}(\hat{\theta}_{k-1}) \to \sigma^2 \sum_{(r,s)\succeq(0,0)} a^2(r,s) + \sum_{i=k}^{P} \frac{(\rho_i^0)^2}{2} \ \text{a.s.} \tag{23}$$

Since $\frac{\log NM}{NM}$ tends to zero, as $\Psi(N,M) \to \infty$, then as $\Psi(N,M) \to \infty$

$$\begin{aligned} &(NM)^{-1}(\chi_\xi(k-1) - \chi_\xi(k)) \\ &\to \log \left(1 + \frac{(\rho_k^0)^2}{2\sigma^2 \sum_{(r,s)\succeq(0,0)} a^2(r,s) + \sum_{i=k+1}^{P} (\rho_i^0)^2} \right) \text{a.s.} \end{aligned} \tag{24}$$

Since $\log \left(1 + \frac{(\rho_k^0)^2}{2\sigma^2 \sum_{(r,s)\succeq(0,0)} a^2(r,s) + \sum_{i=k+1}^{P} (\rho_i^0)^2} \right)$ is strictly positive, then $\chi_\xi(k-1) > \chi_\xi(k)$. Hence, for $k \leq P$, the function $\chi_\xi(k)$ is monotonically decreasing with k.

We next consider the case where $k = P + l$ for any integer $l \geq 1$.

Based on [6], Theorem 1 and Assumption 1 we have that

$$\limsup_{\Psi(N,M)\to\infty} \frac{\sup_{\omega,\upsilon} I_w(\omega,\upsilon)}{\sup_{\omega,\upsilon} f_w(\omega,\upsilon) \log(NM)} \leq 14 \ \text{a.s.} \tag{25}$$

Based on an extension of Theorem 2 we have that a.s. as $\Psi(N,M) \to \infty$

$$\mathcal{L}_{P+l}(\hat{\theta}_{P+l}) = \mathcal{L}_P(\hat{\theta}_P) - \frac{U_l}{NM} + o\left(\frac{\log NM}{NM} \right) \tag{26}$$

where

$$U_l = \sum_{i=1}^{l} I_w(\omega_i, \upsilon_i) \tag{27}$$

is the sum of the l largest elements of the periodogram of the noise field $\{w(s,t)\}$. Clearly

$$U_l \leq l \sup_{\omega,\upsilon} I_u(\omega,\upsilon) \tag{28}$$

Similarly to (16), a.s. as $\Psi(N,M) \to \infty$,

$$\begin{aligned}
&\chi_\xi(P+l) - \chi_\xi(P) \\
&= NM \log \mathcal{L}_{P+l}(\hat{\theta}_{P+l}) + \xi(P+l)\log NM \\
&-NM \log \mathcal{L}_P(\hat{\theta}_P) - \xi P \log NM = \xi l \log NM \\
&+NM \log\left(1 - \frac{U_l}{NM\mathcal{L}_P(\hat{\theta}_P)} + o\left(\frac{\log NM}{NM}\right)\right) \\
&= \xi l \log NM - \left(\frac{U_l}{\mathcal{L}_P(\hat{\theta}_P)} + o(\log NM)\right)(1+o(1)) \\
&= \log NM\left(\xi l - \frac{U_l}{\mathcal{L}_P(\hat{\theta}_P)\log NM} + o(1)\right) \\
&\geq \log NM\left(\xi l - \frac{l \sup\limits_{\omega,\upsilon} I_w(\omega,\upsilon)}{\mathcal{L}_P(\hat{\theta}_P)\log NM} + o(1)\right) \\
&= l \log NM\left(\xi - \frac{\sup\limits_{\omega,\upsilon} I_w(\omega,\upsilon)}{\sup\limits_{\omega,\upsilon} f_w(\omega,\upsilon)\log NM}\frac{\sup\limits_{\omega,\upsilon} f_w(\omega,\upsilon)}{\mathcal{L}_P(\hat{\theta}_P)} + o(1)\right)
\end{aligned} \tag{29}$$

where the second equality is obtained by substituting $\mathcal{L}_{P+l}(\hat{\theta}_{P+l})$ using the equality (26). The third equality is due to the property that for $x \to 0$, $\log(1+x) = x(1+o(1))$, where the observation that the term $\frac{U_l}{NM\mathcal{L}_P(\hat{\theta}_P)}$ tends to zero a.s. as $\Psi(N,M) \to \infty$ is due to (25).

From [5] (or using Theorem 1 in the previous section),

$$\hat{\theta}_P \to \theta_P^0 \quad a.s. \quad \text{as} \quad \Psi(N,M) \to \infty. \tag{30}$$

Hence, the strong consistency (30) of the LSE under the correct model order assumption implies that as $\Psi(N,M) \to \infty$

$$\mathcal{L}_P(\hat{\theta}_P) \to \sigma^2 \sum_{(r,s)\succeq(0,0)} a^2(r,s) \ \text{a.s.} \tag{31}$$

On the other hand using the triangle inequality

$$\sup_{\omega,\upsilon} f_w(\omega,\upsilon) \leq \sigma^2 \sum_{(r,s)\succeq(0,0)} \sum_{(q,t)\succeq(0,0)} |a(r,s)a(q,t)| \tag{32}$$

Substituting (25),(31) and (32) into (29) we conclude that

$$\chi_\xi(P+l) - \chi_\xi(P) > 0 \tag{33}$$

for any integer $l \geq 1$. Therefore, a.s. as $\Psi(N,M) \to \infty$, the function $\chi_\xi(k)$ has a **global minimum** for $k = P$. ■

5. SPECIAL CASE

Introducing some additional restrictions on the structure of the noise field, we can establish a tighter (in terms of ξ) model order selection rule. We thus modify Assumption 1 as follows:

Assumption 1' The stationary noise field $\{w(n,m)\}$ can be represented by infinite order quarter-plane MA noise field, i.e.,

$$w(n,m) = \sum_{r,s=0}^{\infty} a(r,s)u(n-r,m-s) \tag{34}$$

where the field $\{u(n,m)\}$ is an i.i.d. real valued zero-mean random field with finite second order moment, $\sigma^2$and satisfy the condition $E[u(0,0)^2 \log|u(0,0)|] < \infty$. The sequence $a(i,j)$ is a deterministic sequence which satisfied the condition

$$\sum_{r,s=0}^{\infty} (r+s)|a(r,s)| < \infty. \tag{35}$$

In this case based on [1], Theorem 3.2 and Assumption 1' we have that

$$\limsup_{\Psi(N,M)\to\infty} \frac{\sup\limits_{\omega,\upsilon} I_w(\omega,\upsilon)}{\sup\limits_{\omega,\upsilon} f_w(\omega,\upsilon)\log(NM)} \leq 8 \quad \text{a.s.} \tag{36}$$

The results of Theorem 1 and 2 are not affected by this assumption (see [4]). The only change is in Theorem 3. Therefore we can formulate the next theorem:

Theorem 4 *Let Assumptions 1' and 2 be satisfied. Let $\hat{P}$ be given by (13) with $\xi > 8\mathcal{A}$. Then as $\Psi(N,M) \to \infty$*

$$\hat{P} \to P \ \textit{a.s.} \tag{37}$$

The proof of this theorem is identical to the proof of Theorem 3, where instead of (25) we employ the inequality in (36).

6. CONCLUSIONS

We have considered the problem of jointly estimating the number as well as the parameters of two-dimensional sinusoidal signals, observed in the presence of an additive colored noise field. We have established the strong consistency of the LSE when the number of sinusoidal signals is under-estimated, or over -estimated. Based on these results, we have proved the strong consistency of a large family of model order selection rules of the number of sinusoidal components.

7. REFERENCES

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