CRAMER-RAO LOWER BOUND FOR HARMONIC AND SUBHARMONIC ESTIMATION

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ABSTRACT

Recently, Zarowski and Kmpyvnytskyy developed a modified iterative cosinor algorithm (MICA) for the estimation of the parameters of sinusoidal signals with harmonics and subharmonics contaminated by AWGN, and derived the Cramer-Rao Lower Bound (CRLB) for the estimation of fundamental frequency component of such signals. However, their derivation was based on the assumption that the noise variance is known a priori. This paper presents a new derivation of CRLB bound for the case that the noise variance is unknown. The derivations also include the CRLB bounds for the estimation of harmonic and subharmonic amplitudes, noise variance as well as the SNR of the contaminated signal. Numerical simulation results are given to verify and interpret the derived CRLB bounds, together with the evaluation of estimation performance.

1. INTRODUCTION

Parameter estimation for multitone sinusoidal signals in noise has been well reported in the hitherto literature. Typical techniques include the maximum likelihood estimation (MLE) [1], MUSIC [2], and subspace method [3]. However, in these techniques, little attention has been paid to sinusoidal signals with harmonics and subharmonics. In [1], the MLE approach was applied to the general signals consisting of multiple sinusoids in noise under the assumption that there is no special harmonic relationship between the constituent sinusoidal components. Harmonic and subharmonic signal components exist in many signal processing applications. For example, in speech signal processing problems, many acoustic sources such as rotating machinery have non-linear effects within the generating system, and often give rise to harmonics and subharmonics besides the fundamental component. To properly characterize such signals, the harmonics and subharmonics should be taken into account.

Recently, Zarowski and Kmpyvnytskyy [4] developed a modified iterative cosinor algorithm (MICA) for the temperature data containing circadian rhythms for head-injured patients. In the MICA model, the period of the circadian rhythm for the patient, called the patient's "tau" and denoted by T, is to be estimated together with the harmonic and subharmonic components of the data. In [4], the Cramer-Rao Lower Bound (CRLB) for the estimation of the patient's tau was derived but it was assumed that the noise variance is known a priori. This paper presents the derivation of CRLB bounds for the case that the variance is unknown. The derivation also includes the CRLB bounds for other parameters such as amplitudes of harmonics and subharmonics, noise variance, and the signal to noise ratio (SNR). Numerical simulation results are also given to verify and interpret the CRLB bounds.

2. PROBLEM FORMULATION

Let us consider the MICA model for a circadian rhythm data set $\{s(n)\}$ given by

$$s(n) = \sum_{k=1}^{N_H} [A_k \cos(\omega k n) + B_k \sin(\omega k n)] + \sum_{k=2}^{N_S} [C_k \cos(\omega n/k) + D_k \sin(\omega n/k)] + w(n) \quad (1)$$

where $n = 0, 1, \cdots, N-1$, and where N represents the length of the data set. Moreover, $\omega = 2\pi T_s/T \in (0, \pi)$ (T_s represents the sampling period), $A_k, B_k, C_k, D_k \in R$, and $\{w(n)\}$ represents a sequence of independent, identically distributed additive white Gaussian noise (AWGN) random variables with zero mean and an unknown variance of σ^2. Let

$$\begin{aligned}\underline{\alpha} &= [A_1, \cdots, A_{N_H}, B_1, \cdots, B_{N_H}, \\ &\quad C_2, \cdots, C_{N_S}, D_2, \cdots, D_{N_S}]^T \in \mathbf{R}^P \end{aligned} \quad (2)$$

represent the amplitude vector, and let

$$\begin{aligned}\underline{x}[n;T] &= [\cos(\omega n), \cdots, \cos(\omega N_H n), \sin(\omega n), \cdots, \\ &\quad \sin(\omega N_H n), \cos(\omega n/2), \cdots, \cos(\omega n/N_S), \\ &\quad \sin(\omega n/2), \cdots, \sin(\omega n/N_S)]^T \in \mathbf{R}^P \end{aligned} \quad (3)$$

represent the sinusoidal signal vector, where $P = 2(N_H + N_S - 1)$. Then, (1) can be rewritten compactly as

$$s(n) = \underline{x}^T [n;T] \, \underline{\alpha} + w(n). \quad (4)$$

Moreover, let

$$\underline{\theta} = \left[\underline{\alpha}^T \; T \; \sigma^2\right]^T = [\theta_1, \theta_2, \cdots, \theta_P, \; \theta_{P+1}, \theta_{P+2}]^T \quad (5)$$

represent the parameter vector, where $[\theta_1, \theta_2, \cdots, \theta_P] = \underline{\alpha}^T$, $\theta_{P+1} = T$, and $\theta_{P+2} = \sigma^2$. Then, the problem under consideration amounts to estimating $\underline{\theta}$ from the N-point noisy data $\underline{s} = [s(0), s(1), \cdots, s(N-1)]^T \in \mathbf{R}^N$. The pdf of $\underline{s}$ is given by

$$p(\underline{s};\underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left[s(n) - \underline{x}^T[n;T]\,\underline{\alpha}\right]^2\right\}. \tag{6}$$

It is convenient to define the matrices

$$\begin{aligned} X(T) &= [\underline{x}[0;T], \underline{x}[1;T], \ \cdots, \underline{x}[N-1;T]]^T \in \mathbf{R}^{N\times P}, \\ A(T) &= X^T(T)\,X(T) \in \mathbf{R}^{P\times P}, \end{aligned} \tag{7}$$

together with $\rho = \underline{s}^T\underline{s} \in \mathbf{R}$, and $\underline{g}(T) = X^T(T)^1\underline{s} \in \mathbf{R}^P$. Consequently, (6) may be rewritten as a log likelihood function as[2]

$$\begin{aligned} \ln p(\underline{s};\underline{\theta}) &= \ln\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} - \frac{1}{2\sigma^2}(\underline{s} - X\underline{\alpha})^T(\underline{s} - X\underline{\alpha}) \\ &= -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left(\underline{\alpha}^T A\underline{\alpha} - 2\underline{g}^T\underline{\alpha} + \rho\right). \end{aligned} \tag{8}$$

The data model developed above will be used for the derivation of the desired CRLB bounds.

3. THE DERIVATION OF CRLB BOUNDS

For the CRLB bounds to exist, the regularity condition

$$E\left[\frac{\partial \ln p(\underline{s};\underline{\theta})}{\partial\underline{\theta}}\right] = \mathbf{0} \tag{9}$$

must be satisfied [5]. By long but straightforward manipulations, it can be shown that

$$E\left[\frac{\partial \ln p(\underline{s};\underline{\theta})}{\partial\underline{\alpha}}\right] = \frac{1}{\sigma^2}E\left[\underline{g} - A\underline{\alpha}\right] = 0. \tag{10}$$

$$E\left[\frac{\partial \ln p(\underline{s};\underline{\theta})}{\partial T}\right] = E\left[\frac{1}{\sigma^2}\frac{\partial \underline{g}^T}{\partial T}\underline{\alpha} - \frac{1}{2\sigma^2}\underline{\alpha}^T\frac{\partial A}{\partial T}\underline{\alpha}\right] = 0, \tag{11}$$

$$E\left[\frac{\partial \ln p(\underline{s};\underline{\theta})}{\partial(\sigma^2)}\right] = -\frac{N}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}E\left[\underline{\alpha}^T A\underline{\alpha} - 2\underline{g}^T\underline{\alpha} + \rho\right] = 0. \tag{12}$$

By combining (10)-(12), one can readily arrive at (9).

The Fisher information matrix (FIM) is defined as

$$I(\underline{\theta}) = \{I_{ij},\ i,j = 1,2,3\}$$

$$= E\left\{\begin{bmatrix} \dfrac{\partial \ln p}{\partial\underline{\alpha}} \\ \dfrac{\partial \ln p}{\partial T} \\ \dfrac{\partial \ln p}{\partial\sigma^2}\end{bmatrix}\begin{bmatrix}\left(\dfrac{\partial \ln p}{\partial\underline{\alpha}}\right)^T & \dfrac{\partial \ln p}{\partial T} & \dfrac{\partial \ln p}{\partial\sigma^2}\end{bmatrix}\right\},$$

where

$$\begin{aligned} I_{11} &= E\left[\frac{\partial \ln p}{\partial\underline{\alpha}}\left(\frac{\partial \ln p}{\partial\underline{\alpha}}\right)^T\right] \\ &= \frac{1}{\sigma^4}E\left[(\underline{g} - A\underline{\alpha})(\underline{g} - A\underline{\alpha})^T\right] = \frac{1}{\sigma^2}X^T X. \end{aligned} \tag{13}$$

Similarly, by mathematical manipulations, one can get

$$I_{12} = \frac{1}{\sigma^2}X^T\frac{\partial X}{\partial T}\underline{\alpha},\ \ I_{22} = \frac{1}{\sigma^2}\underline{\alpha}^T\frac{\partial X^T}{\partial T}\frac{\partial X}{\partial T}\underline{\alpha},$$

$$I_{21} = I_{12}^T,\ I_{31} = I_{13}^T = 0,\ I_{32} = I_{23}^T = 0,\ \ I_{33} = \frac{N}{2\sigma^4}.$$

Consequently, the FIM is obtained as

$$I(\underline{\theta}) = \frac{1}{\sigma^2}\begin{bmatrix} X^T X & X^T\dfrac{\partial X}{\partial T}\underline{\alpha} & 0 \\ \underline{\alpha}^T\dfrac{\partial X^T}{\partial T}X & \underline{\alpha}^T\dfrac{\partial X^T}{\partial T}\dfrac{\partial X}{\partial T}\underline{\alpha} & 0 \\ 0 & 0 & \dfrac{N}{2\sigma^2}\end{bmatrix}.$$

Let the matrix P be defined as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix} = \frac{1}{\sigma^2}\begin{bmatrix} X^T X & X^T\dfrac{\partial X}{\partial T}\underline{\alpha} \\ \underline{\alpha}^T\dfrac{\partial X^T}{\partial T}X & \underline{\alpha}^T\dfrac{\partial X^T}{\partial T}\dfrac{\partial X}{\partial T}\underline{\alpha}\end{bmatrix}. \tag{14}$$

Then, the CRLB bound is obtained as the inverse of FIM in accordance with

$$I^{-1}(\underline{\theta}) = \begin{bmatrix} P^{-1} & 0 \\ 0 & \dfrac{2\sigma^4}{N}\end{bmatrix}, \tag{15}$$

where

$$P^{-1} = \left[\begin{array}{c|c} \left(P_{11} - P_{12}P_{22}^{-1}P_{21}\right)^{-1} & -\left(P_{11} - P_{12}P_{22}^{-1}P_{21}\right)^{-1}P_{12}P_{22}^{-1} \\ -\left(P_{22} - P_{21}P_{11}^{-1}P_{12}\right)^{-1}P_{21}P_{11}^{-1} & \left(P_{22} - P_{21}P_{11}^{-1}P_{12}\right)^{-1}\end{array}\right],$$

and where P_{11} and P_{22} are assumed to be invertible.

In this way, the CRLB bound on the variance of the unbiased estimator of T is given by

$$\begin{aligned} U &= \left(P_{22} - P_{21}P_{11}^{-1}P_{12}\right)^{-1} \\ &= \sigma^2\left[\underline{y}^T\left[I - X\left(X^T X\right)^{-1}X^T\right]\underline{y}\right]^{-1}, \end{aligned} \tag{16}$$

[1]Here, T is used in two different contexts, as a superscript to denote matrix transposition, and as the signal period.

[2]In the following, signal dependence on T is omitted from the notations for the sake of simplicity, e.g. $\underline{g}$ stands for $\underline{g}(T)$, X for $X(T)$, and A for $A(T)$, etc.

where $\underline{y} = \frac{\partial X}{\partial T}\underline{\alpha} \in \mathbf{R}^N$ and I is a $N \times N$ identity matrix.

The structure of (16) has a geometric interpretation. From [6], if $M < N$ for a matrix $H \in \mathbf{R}^{N\times M}$, where H is of rank M, then the matrix operator

$$P_H = H\left(H^T H\right)^{-1} H^T \in \mathbf{R}^{N\times N} \qquad (17)$$

is an orthogonal projection matrix (i.e., it is idempotent[3] and symmetric). This implies that the vector $v \in \mathbf{R}^N$ is projected into the subspace of vector space $\mathbf{R}^N$ spanned by the columns of H under the operation $P_H v$.

In (16), let the matrix $P_X^\perp$ be defined as

$$P_X^\perp = I - X\left(X^T X\right)^{-1} X^T \in \mathbf{R}^{N\times N} \qquad (18)$$

which is an orthogonal projection matrix. Then,

$$U = \sigma^2 \left[\underline{y}^T P_X^\perp \underline{y}\right]^{-1} = \sigma^2 \left[\left(P_X^\perp \underline{y}\right)^T P_X^\perp \underline{y}\right]^{-1}. \qquad (19)$$

In (19), $P_X^\perp \underline{y} \in \mathbf{R}^N$ is a column vector of length N. In this way, $\left(P_X^\perp \underline{y}\right)^T P_X^\perp \underline{y}$ can be considered as the energy of $P_X^\perp \underline{y}$. Therefore, U is a ratio of the noise power σ^2 to the signal power, i.e. it is the inverse of SNR. This implies that the higher the SNR, the lower the CRLB bound.

The CRLB bounds on the estimation variances of other parameters can be derived from (15). For the estimation $\hat{\underline{\alpha}}$ of the amplitude vector $\underline{\alpha}$, the CRLB bound can be obtained as

$$\begin{aligned} V &= \left(P_{11} - P_{12}P_{22}^{-1}P_{21}\right)^{-1} \\ &= \sigma^2 \left[X^T \left[I - \underline{y}\left(\underline{y}^T \underline{y}\right)^{-1} \underline{y}^T\right] X\right]^{-1} \in \mathbf{R}^{P\times P}. \end{aligned} \qquad (20)$$

The CRLB bound for the estimation of the noise power σ^2 is simply given by

$$W = \frac{2\sigma^4}{N}. \qquad (21)$$

Finally, one can also derive the CRLB bound for estimating the SNR of the data. According to [5], the CRLB bound for an estimator $\beta = g(\underline{\theta})$ is given by

$$S = \frac{\partial g}{\partial \underline{\theta}} I^{-1}(\underline{\theta}) \left(\frac{\partial g}{\partial \underline{\theta}}\right)^T.$$

Here, the SNR for the MICA signal model is defined as the ratio of the sum of fundamental frequency component power to the noise power in accordance with

$$\beta = h(\underline{\theta}) = \frac{A_1^2 + B_1^2}{\sigma^2}.$$

Therefore, the CRLB bound for the SNR estimation is obtained as

$$S = \frac{\partial h}{\partial \underline{\theta}} I^{-1}(\underline{\theta}) \left(\frac{\partial h}{\partial \underline{\theta}}\right)^T, \qquad (22)$$

[3] A matrix operator P is idempotent if $P^2 = P$

where

$$\frac{\partial h}{\partial \underline{\theta}} = \left[\underbrace{\frac{2A_1}{\sigma^2}\cdots 0}_{N_H}\ \underbrace{\frac{2B_1}{\sigma^2}\cdots 0}_{N_H}\ \underbrace{0\cdots 0}_{N_S-1}\ \underbrace{0\cdots 0}_{N_S-1}\ 0\ \frac{-A_1^2 - B_1^2}{(\sigma^2)^2}\right].$$

The above derived CRLB bounds are useful in estimating how many data points need to be collected in order to achieve a desired accuracy under various modelling conditions. The derived CRLB bounds will be verified by computer simulation in the next section.

4. THE SIMULATION RESULTS

In this section, the validity of the CRLB bounds in (16) and (19)-(22) is confirmed through computer simulations. This is achieved by the Monte Carlo simulations for the above parameters using MLE algorithm. The simulations are based on $\underline{\alpha} = [1\ 0.2\ 0.05\ 0.01\ 1\ 0.2\ 0.05\ 0.01\ 4\ 6\ 4\ 6]^T$ in (2). The data size is chosen as $N = 500$, as during the simulations, if the data size is too small (e.g. $N = 100$), the condition number of the matrix A in (7) becomes very large, rendering A close to singular. This makes the estimation less accurate or even unreliable. Therefore, a data size N of 500 is used to obtain the desired estimation accuracy. For each SNR point, 500 independent Monte Carlo simulations are made and the results are averaged to arrive at an estimation.

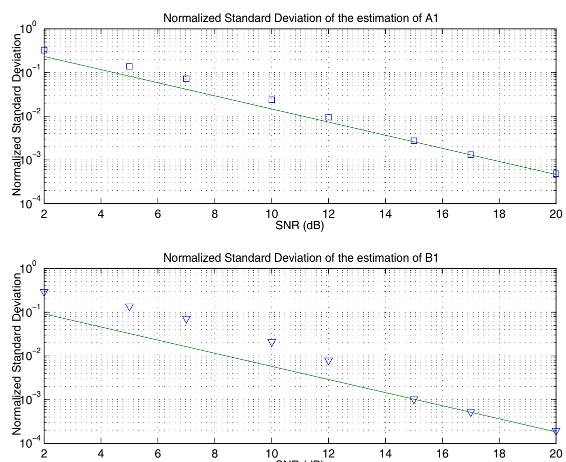

Fig. 1. The normalized standard deviation of the estimation of $A1$ (top) and $B1$ (bottom) vervus SNRs together with the CRLBs (solid curves).

The estimation variances for parameters $A1$, $B1$ are plotted versus SNR (in dB) in Fig. 1, the estimation variance of σ^2 and SNR are plotted in Fig. 2, and the estimation variance of T is plotted in Fig. 3. It is observed that the simulated variance approaches the CRLB bound as SNR increases (as

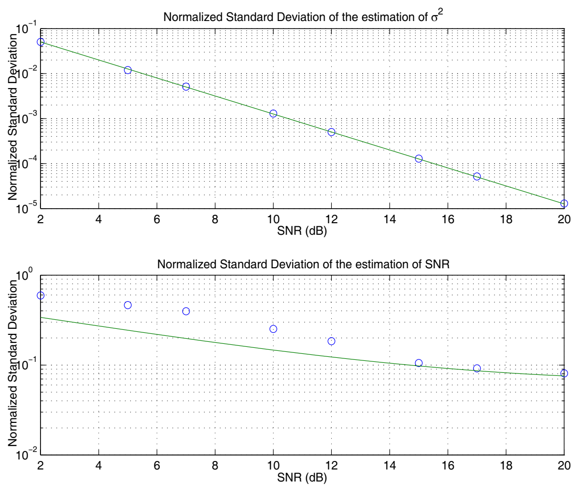

Fig. 2. The normalized standard deviation of the estimation of σ^2 (top) and SNR (bottom) versus SNRs together with the CRLBs (solid curves).

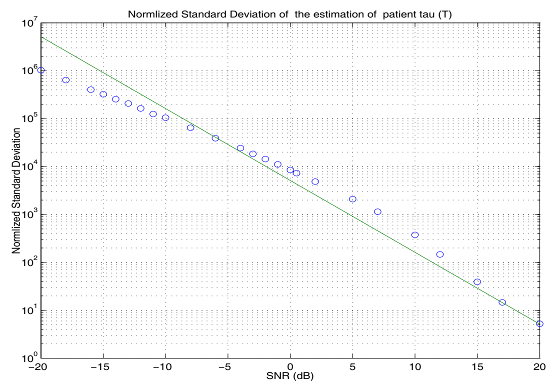

Fig. 3. The normlized standard deviation of the estimation of T together with CRLB (solid curve).

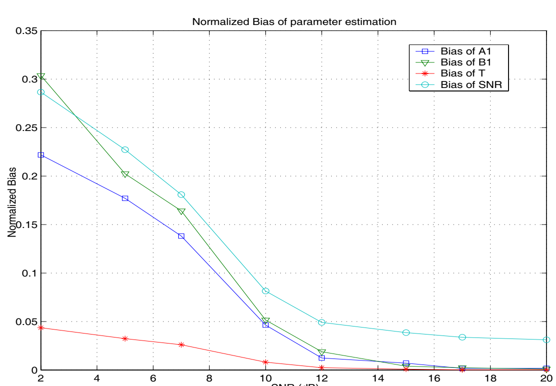

Fig. 4. The estimation biases versus SNRs.

expected). Since CRLB bound is the theoretical limit on the best possible unbiased estimation performance, no other estimation method can be expected to perform better than MLE at high SNRs. With decreasing SNR, the estimation variance becomes worse than the bound (again, as expected). The simulation results verify the validity of the derived CRLB bounds. However, at SNRs smaller than -5 dB, the simulation estimation variance for T does not conform to the bound (c.f. Fig. 3), simply because the CRLB bound is valid for unbiased estimation only. At very low SNRs, the estimator tends to be biased as is obvious from Fig. 4, and the unbiased CRLB bound is not the valid bound anymore. In this case, the biased CRLB bound should be considered instead. For the simulation results, the unbiased CRLB bound is a very good predictor of the true variance for SNRs greater than about 14 or 15 dB.

5. CONCLUSION

This paper has presented CRLB bounds for the unbiased estimation of parameters in circadian rhythm data model MICA under the assumption that noise variance is not known a priori. The resulting CRLB bounds have been confirmed through Monte Carlo simulations using the MLE estimation method. It has been observed that the estimators achieve the CRLB bound at high SNRs, but that at very low SNRs, the estimators tend to be biased and therefore the unbiased CRLB bound is no longer the valid bound. In this way, future work involves the study of the biased CRLB bound in MICA model.

6. REFERENCES

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