FREQUENCY ESTIMATION USING TAPERED DATA

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ABSTRACT

The maximizer of the periodogram of a sinusoid in additive noise is known to have optimal asymptotic properties even when the noise is neither Gaussian nor white. The effect of tapering or windowing on the accuracy of the estimator does not appear to have been considered previously. In this paper, we present the asymptotic theory for the maximizer of windowed periodograms of Hamming and Hanning-type. We also introduce and analyse two closed-form frequency estimators constructed from three Fourier coefficients of a Hanning-tapered process.

1. INTRODUCTION

The statistical properties of various estimators of the frequency of a noisy sinusoid are well-understood [1]. In particular, the maximizer of the periodogram has the same asymptotic properties as the maximum likelihood estimator constructed under Gaussian white assumptions, even when the underlying noise process is neither Gaussian nor white. Routinely in signal processing it is not the raw time series which is analyzed, but a tapered version of it. This tapering is mainly done to obtain smoother and more accurate estimates of the underlying noise spectral density, but there appears to have been no theory developed to assess the effects of tapering on the maximizer of the periodogram of the tapered data. Note that estimating the spectral density and the location of a sinusoidal frequency are totally different problems.

Quinn [2][3] and MacLeod [4] have considered the use of Fourier coefficients at three neighbouring frequencies to obtain accurate estimators of frequency. These techniques work, however, only when the Fourier coefficients of the untapered time series are used.

In this paper, we develop asymptotic theory for the maximizer of the periodogram of a noisy sinusoid, for a class of cosine tapers which includes the Hanning and Hamming tapers. We also propose and develop asymptotic theory for estimators based on the use of Fourier coefficients of Hanning-tapered series at adjacent frequencies, including two three-Fourier-coefficient estimators based on the Quinn [2][3] estimators. Note that it is impossible to derive any theoretical results for fixed sample size for any frequency estimator of this type.

2. ASYMPTOTIC THEORY OF THE WINDOWED PERIODOGRAM MAXIMISER

In what follows we shall assume that $\{X_t\}$ is a discrete time stochastic process satisfying an equation of the form

$$X_t = \rho \cos(\omega_0 t + \phi) + \varepsilon_t,\ t = 0, 1, \ldots$$

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where ρ, ω_0 and ϕ are unknown constants, with $0 < \omega_0 < \pi$, and $\{\varepsilon_t\}$ is a stationary, zero mean, ergodic stochastic process satisfying the conditions given in [2]. Note that $\{\varepsilon_t\}$ need not be Gaussian or white. Let $J_X(\omega) = \sum_{t=0}^{T-1} e^{-i\omega t} X_t$.The periodogram $I_X(\omega)$ of $\{X_0, X_1, \ldots, X_{T-1}\}$ is defined by $I_X(\omega) = 2T^{-1} |J_X(\omega)|^2$. Its maximizer, $\widehat{\omega}_T$, satisfies $T(\widehat{\omega}_T - \omega_0) \to 0$ almost surely as $T \to \infty$ [5], while the distribution of $T^{3/2}(\widehat{\omega}_T - \omega_0)$ converges to the normal with mean 0 and variance $48\pi f(\omega_0)/\rho^2$, where $f(\omega) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} \gamma_j e^{-ij\omega}$ is the spectral density of $\{\varepsilon_t\}$ and $\gamma_j = E(\varepsilon_t \varepsilon_{t-j})$. Let $Y_t = h_t X_t$, where $h_t = \alpha - 2\beta \cos(at/T)$ and α and β are suitably chosen constants. Let $\widehat{\omega}_T$ maximize $I_Y(\omega) = 2T^{-1} |J_Y(\omega)|^2$, where $J_Y(\omega) = \sum_{t=0}^{T-1} e^{-i\omega t} Y_t$. Then

Theorem 1 $T^{3/2}(\widehat{\omega}_T - \omega_0)$ *is asymptotically normal with mean* 0 *and variance* $32\pi f(\omega_0) c_3/c_4^2$, *where* c_3 *and* c_4 *are given by* (5) *and* (6) , *if* $g(x)$, *given by* (3) , *has global maximum at* $x = 0$.

Proof.

$$\begin{aligned} J_Y(\omega) &= \sum_{t=0}^{T-1} e^{-i\omega t} \left\{ \alpha - \beta e^{iat/T} - \beta e^{-iat/T} \right\} X_t \qquad (1) \\ &= \alpha J_X(\omega) - \beta J_X(\omega - a/T) - \beta J_X(\omega + a/T). \end{aligned}$$

Now, letting $D = \rho e^{i\phi}/2$, and $U_T(\omega) = \sum_{t=0}^{T-1} e^{-i\omega t} \varepsilon_t$, we have

$$J_X(\omega) = D S_T(\omega - \omega_0) + \overline{D} S_T(\omega + \omega_0) + U_T(\omega) \qquad (2)$$

where $S_T(\lambda) = \sum_{t=0}^{T-1} e^{-i\lambda t}$ is $\left(1 - e^{-i\lambda T}\right) / \left(1 - e^{-i\lambda}\right)$ if $\lambda \not\equiv 0 \bmod (2\pi)$ and T otherwise. Thus, $J_Y(\omega) = D Z_T(\omega) + V_T(\omega) + O(1)$, for $\omega \in (0, \pi)$, almost surely as $T \to \infty$, where

$$\begin{aligned} Z_T(\omega) &= \alpha S_T(\omega - \omega_0) - \beta S_T(\omega - \omega_0 - a/T) \\ &\quad - \beta S_T(\omega - \omega_0 + a/T) \\ V_T(\omega) &= \alpha U_T(\omega) - \beta U_T(\omega - a/T) - \beta U_T(\omega + a/T). \end{aligned}$$

Note that $S_T(\omega + \omega_0)$ is $O(1)$ since $0 < \omega + \omega_0 < 2\pi$. From [5] and [6] it follows that $V_T(\omega) = O_P\left(T^{1/2}\right)$ and $V_T(\omega) = O_{a.s.}\left(\{T \log T\}^{1/2}\right)$, uniformly in ω. However,

$$\begin{aligned} Z_T(\omega_0) &= \alpha T - \beta \frac{1 - e^{ia}}{1 - e^{ia/T}} - \beta \frac{1 - e^{-ia}}{1 - e^{-ia/T}} \\ &= T\left(\alpha - 2\beta \frac{\sin a}{a}\right) + O(1) \end{aligned}$$

and if $\omega \neq \omega_0$, $Z_T(\omega)$ is $O(1)$. It is therefore plausible that $I_Y(\omega)$ is maximized in a neighbourhood of ω_0, unless $\alpha - 2\beta \frac{\sin a}{a} = 0$, which we shall exclude. Now

$$\begin{aligned} &Z_T\left(\omega_0 + \frac{x}{T}\right) \\ &= T\left\{ \alpha \frac{1 - e^{-ix}}{ix} - \beta \frac{1 - e^{-i(x-a)}}{i(x-a)} - \beta \frac{1 - e^{-i(x+a)}}{i(x+a)} \right\} + O(1) \end{aligned}$$

and thus, since $T^{-1}V_T(\omega) \to 0$ almost surely and uniformly in ω, $T^{-1}I_Y\left(\omega_0 + \frac{x}{T}\right)$ converges to

$$g(x) = 2\left|\alpha\frac{1-e^{-ix}}{x} - \beta\frac{1-e^{-i(x-a)}}{x-a} - \beta\frac{1-e^{-i(x+a)}}{x+a}\right|^2, \tag{3}$$

almost surely as $T \to \infty$. A necessary condition that $T(\widehat{\omega}_T - \omega_0)$ converge almost surely to 0 is that therefore that $g(x)$ have global maximum at $x = 0$ and the proof in [5] can be adapted to show sufficiency. Now

$$0 = I'_Y(\widehat{\omega}_T) = I'_Y(\omega_0) + I''_Y(\widetilde{\omega}_T)(\widehat{\omega}_T - \omega_0), \tag{4}$$

where $\widetilde{\omega}_T$ is between $\widehat{\omega}_T$ and ω_0. $T(\widetilde{\omega}_T - \omega_0) \to 0$, almost surely, since $T(\widehat{\omega}_T - \omega_0)$ does. The central limit theorem follows from that for $T^{-3/2}I'_Y(\omega_0)$ and by showing that $T^{-3}I''_Y(\omega)$ converges almost surely in a neighbourhood of ω_0. Now

$$\begin{aligned} I'_Y(\omega) &= \frac{4}{T}\operatorname{Re}\left\{\overline{J}_Y(\omega)J'_Y(\omega)\right\}, \\ J'_Y(\omega) &= DZ'_T(\omega) + V'_T(\omega) + O(T), \end{aligned}$$

since when $0 < \lambda < \pi$, $S'_T(\lambda) = \sum_{t=0}^{T-1} -ite^{-i\lambda t} = O(T)$. Also, since the real and imaginary parts of

$$T^{-3/2}U'_T(\omega) = T^{-3/2}\sum_{t=0}^{T-1} -ite^{-i\omega t}\varepsilon_t$$

are asymptotically normal with finite variances,

$$V'_T(\omega) = \alpha U'_T(\omega) - \beta U'_T(\omega - a/T) - \beta U'_T(\omega + a/T),$$

is $O_P\left(T^{3/2}\right)$. Finally,

$$\begin{aligned} Z'_T(\omega) &= \alpha S'_T(\omega - \omega_0) - \beta S'_T(\omega - \omega_0 - a/T) \\ &\quad -\beta S'_T(\omega - \omega_0 + a/T), \end{aligned}$$

which will be shown to be $O\left(T^2\right)$ when $\omega = \omega_0$. Thus

$$T^{-3/2}J'_Y(\omega_0) = DT^{-3/2}Z'_T(\omega_0) + T^{-3/2}V'_T(\omega_0) + O\left(T^{-1/2}\right).$$

Now, when $0 < \lambda < \pi$, $S'_T(\lambda)$ is

$$-iTe^{-i\lambda T}\left(e^{-i\lambda} - 1\right) + i\left(e^{-i\lambda T} - 1\right)e^{-i\lambda}/\left(e^{-i\lambda} - 1\right)^2,$$

and so when $a \neq 0$,

$$S'_T(a/T) = T^2\frac{e^{-ia}}{a} - iT^2\frac{e^{-ia} - 1}{a^2} + O(T).$$

Also, if $0 < \lambda < \pi$,

$$\begin{aligned} S''_T(\lambda) &= -T^2\frac{e^{-i\lambda T}}{e^{-i\lambda} - 1} + 2T\frac{e^{-i\lambda T}e^{-i\lambda}}{\left(e^{-i\lambda} - 1\right)^2} \\ &\quad + \frac{\left(e^{-i\lambda T} - 1\right)e^{-i\lambda}}{\left(e^{-i\lambda} - 1\right)^2} - 2\frac{\left(e^{-i\lambda T} - 1\right)e^{-2i\lambda}}{\left(e^{-i\lambda} - 1\right)^3}, \end{aligned}$$

while $S''_T(0) = -T^3/3 + O\left(T^2\right)$, and so, when $a \neq 0$,

$$S''_T(a/T) = T^3\left(-i\frac{e^{-ia}}{a} - 2\frac{e^{-ia}}{a^2} + 2i\frac{e^{-ia} - 1}{a^3}\right) + O\left(T^2\right).$$

Thus

$$Z'_T(\omega_0) = -iT^2\left\{\alpha/2 - 2\beta\left(\frac{\sin a}{a} + \frac{\cos a - 1}{a^2}\right)\right\} + O(T).$$

Consequently $T^{-3/2}I'_Y(\omega_0)$ equals

$$4T^{-5/2}\operatorname{Re}\left(\overline{D}\left\{Tc_1V'_T(\omega_0) + iT^2c_2V_T(\omega_0)\right\}\right) + o_P(1),$$

where $c_1 = \alpha - 2\beta\sin a/a$ and

$$c_2 = \alpha/2 - 2\beta\left\{\sin a/a + (\cos a - 1)/a^2\right\}.$$

Now

$$\begin{aligned} &Tc_1V'_T(\omega_0) + iT^2c_2V_T(\omega_0) \\ &= Tc_1\left\{\alpha U'_T(\omega_0) - \beta U'_T(\omega_0 - a/T) - \beta U'_T(\omega_0 + a/T)\right\} \\ &+ iT^2c_2\left\{\alpha U_T(\omega_0) - \beta U_T(\omega_0 - a/T) - \beta U_T(\omega_0 + a/T)\right\} \\ &= -iT\sum_{t=0}^{T-1}(c_1t - c_2T)(b_{t,T} + ia_{t,T})\varepsilon_t, \end{aligned}$$

where $b_{t,T} + ia_{t,T} = e^{-i\omega_0 t}\left\{\alpha - 2\beta\cos(at/T)\right\}$. Thus, to $o_P(1)$,

$$T^{-3/2}I'_Y(\omega_0) = -4T^{-3/2}\sum_{t=0}^{T-1}(c_1t - c_2T)(a_{t,T}D_r + b_{t,T}D_i)\varepsilon_t$$

where $D_r + iD_i = D$. We next use [6], to show that $T^{-3/2}I'_Y(\omega_0)$ is asymptotically normal with mean 0 and variance $32\pi f(\omega_0)c_3$, where

$$\begin{aligned} c_3 &= \lim_{T\to\infty} T^{-3}\sum_{t=0}^{T-1}(c_1t - c_2T)^2(a_{t,T}D_r + b_{t,T}D_i)^2 \\ &= \frac{|D|^2}{2}\left(\left(\alpha^2 + 2\beta^2\right)\left(\frac{c_1^2}{3} - c_1c_2 + c_2^2\right)\right. \\ &\quad - 4\alpha\beta\left\{c_1^2\tau_1(a) - 2c_1c_2\tau_2(a) + c_2^2\frac{\sin a}{a}\right\} \\ &\quad \left.+2\beta^2\left[c_1^2\tau_1(2a) - 2c_1c_2\tau_2(2a) + c_2^2\frac{\sin(2a)}{2a}\right]\right), \end{aligned} \tag{5}$$

$$\begin{aligned} \tau_1(a) &= \frac{\sin a}{a} + \frac{2\cos a}{a^2} - \frac{2\sin a}{a^3}, \\ \tau_2(a) &= \frac{\sin a}{a} - \frac{1 - \cos a}{a^2}. \end{aligned}$$

Now

$$\begin{aligned} I''_Y(\omega) &= \frac{4}{T}\operatorname{Re}\left\{\left|J'_T(\omega)\right|^2 + \overline{J}_T(\omega)J''_T(\omega)\right\}, \\ J''_T(\omega) &= DZ''_T(\omega) + V''_T(\omega) + O\left(T^2\right). \end{aligned}$$

But, $V''_T(\omega) = O_{a.s.}\left(T^{5/2}(\log\log T)^{1/2}\right)$ uniformly in ω. Also

$$\begin{aligned} T^{-3}Z''_T(\widetilde{\omega}_T) &= T^{-3}\left\{\alpha S''_T(\widetilde{\omega}_T - \omega_0)\right. \\ &\quad \left.-\beta S''_T(\widetilde{\omega}_T - \omega_0 - a/T) - \beta S''_T(\widetilde{\omega}_T - \omega_0 + a/T)\right\}, \end{aligned}$$

which, since $T(\widetilde{\omega}_T - \omega_0)$ converges almost surely to 0, converges almost surely to

$$\lim_{T\to\infty} T^{-3}Z''_T(\omega_0) = -\frac{\alpha}{3} + 2\beta\tau_1(a).$$

Thus $T^{-3}J_T''(\widetilde{\omega}_T) \to D\left\{-\frac{\alpha}{3}+2\beta\tau_1(a)\right\}$ almost surely and, since $T^{-2}J_T'(\widetilde{\omega}_T) \to -iD\left\{\frac{\alpha}{2}-2\beta\tau_2(a)\right\}$ and $T^{-1}J_T(\widetilde{\omega}_T) \to \alpha - 2\beta\frac{\sin a}{a}$, it follows that $T^{-3}I_Y(\widetilde{\omega}_T)$ converges almost surely to

$$\begin{aligned} c_4 &= 4|D|^2\left|\frac{\alpha}{2}-2\beta\tau_2(a)\right|^2 \\ &\quad +4|D|^2\left(\alpha-2\beta\frac{\sin a}{a}\right)\left\{-\frac{\alpha}{3}+2\beta\tau_1(a)\right\}. \quad (6)\end{aligned}$$

Finally, since, from (4), $\widehat{\omega}_T-\omega_0 = -I_Y'(\omega_0)/I_Y''(\widetilde{\omega}_T)$, the result follows. ■

The main interest is the case where $a = 2\pi$, corresponding to the Hanning and Hamming tapers. We then have

$$c_1=\alpha, c_2=\alpha/2, c_4=|D|^2\alpha\left(-\frac{\alpha}{3}+\frac{4\beta}{\pi^2}\right)$$
$$c_3=|D|^2\alpha^2\left\{\frac{\alpha^2}{24}-\frac{\alpha\beta}{\pi^2}+\beta^2\left(\frac{1}{12}+\frac{1}{8\pi^2}\right)\right\}.$$

Thus the asymptotic variance when $a=2\pi$ is

$$128\pi f(\omega_0)\frac{\frac{\alpha^2}{24}-\frac{\alpha\beta}{\pi^2}+\beta^2\left(\frac{1}{12}+\frac{1}{8\pi^2}\right)}{\rho^2\left(-\frac{\alpha}{3}+\frac{4\beta}{\pi^2}\right)^2}.$$

When $\beta=0$, i.e. when there is no tapering, this is just

$$v=\frac{48\pi f(\omega_0)}{\rho^2},$$

in agreement with [5]. For the Hanning and Hamming tapers, the asymptotic variances are respectively $2.3428v$ and $1.6444v$.

3. HANNING 3-POINT ESTIMATORS

We assume from now on that $a = 2\pi$. As in [2][3], we wish to construct an estimator of ω_0 using only a small number of Fourier coefficients. In this case, we use only the tapered Fourier coefficients, $J_Y(\omega_j)$, where $\omega_j = 2\pi j/T$. From (1) and (2), $J_Y(\omega_j)$ is

$$\begin{aligned} &D\{\alpha S_T(\omega_j-\omega_0)-\beta S_T(\omega_{j-1}-\omega_0)-\beta S_T(\omega_{j+1}-\omega_0)\} \\ &\quad +\alpha U_T(\omega_j)-\beta U_T(\omega_{j-1})-\beta U_T(\omega_{j+1})+O(1).\end{aligned}$$

We now use the ideas developed in [2][3]. Define n_T, δ_T and $\widehat{n}_T$, possibly not uniquely, by $\omega_0 = 2\pi T^{-1}(n_T+\delta_T)$, $|\delta_T| \le 1/2$ and

$$\widehat{n}_T = \underset{2\le j\le\lfloor (T-3)/2\rfloor}{\operatorname{argmax}} |J_Y(\omega_j)|^2.$$

Consider using only $\{J_Y(\omega_j); j = n_T-1, n_T, n_T+1\}$ to estimate ω_0. Although we cannot know n_T, we shall use $\widehat{n}_T$ to estimate it, which has no asymptotic effect. Since the real and imaginary parts of

$$\left\{T^{-1/2}U_T(\omega_j); j=n_T-2,\dots,n_T+2\right\}$$

are asymptotically independent and normal with means 0 and variances $\sigma^2 = 2\pi f(\omega_0)$ [6], it follows that

$$\{J_Y(\omega_j); j=n_T-1,n_T,n_T+1\}$$

is a set of asymptotically complex normal (dependent) random variables. The asymptotic log-likelihood constructed from these is

$$-\frac{1}{2T\sigma^2}(J-DZ)^*\Sigma^{-1}(J-DZ),$$

apart from an additive constant involving σ^2, where

$$J=\begin{bmatrix} J_Y(\omega_{n_T-1}) & J_Y(\omega_{n_T}) & J_Y(\omega_{n_T+1})\end{bmatrix}',$$
$$Z=\begin{bmatrix} Z_T(\omega_{n_T-1}) & Z_T(\omega_{n_T}) & Z_T(\omega_{n_T+1})\end{bmatrix}'$$

and

$$\Sigma=\begin{bmatrix}\alpha^2+2\beta^2 & -2\alpha\beta & \beta^2\\ -2\alpha\beta & \alpha^2+2\beta^2 & -2\alpha\beta \\ \beta^2 & -2\alpha\beta & \alpha^2+2\beta^2\end{bmatrix},$$

the latter form calculated using the asymptotic dependence structure of the $V_T(\omega_j)$.

Now, for $j=-1,0,1$, $DZ_T(\omega_{n_T+j})$ is asymptotically

$$TD\frac{e^{i2\pi\delta}-1}{2\pi i\delta}\left(\alpha\frac{\delta}{\delta-j}-\beta\frac{\delta}{\delta+1-j}-\beta\frac{\delta}{\delta-1-j}\right)$$

or $Ed_j(\delta)$, say, where $E = TD\left(e^{i2\pi\delta}-1\right)/(2\pi i\delta)$. Let

$$d(\delta)=\begin{bmatrix} d_{-1}(\delta) & d_0(\delta) & d_1(\delta)\end{bmatrix}'.$$

We could thus estimate δ by minimizing with respect to δ

$$\begin{aligned} S(\delta) &= \min_E (J-Ed(\delta))^*\Sigma^{-1}(J-Ed(\delta)) \quad (7)\\ &= J^*\Omega J - \frac{|d'(\delta)\Omega J|^2}{d'(\delta)\Omega d(\delta)},\end{aligned}$$

where $\Omega=\Sigma^{-1}$ is given by

$$\Omega_{11}=\Omega_{33}=\frac{\alpha^4+4\beta^4}{(\alpha^2+\beta^2)(\alpha^4-3\alpha^2\beta^2+6\beta^4)}$$
$$\Omega_{12}=\Omega_{23}=\Omega_{21}=\Omega_{32}=\frac{2\alpha\beta}{\alpha^4-3\alpha^2\beta^2+6\beta^4}$$
$$\Omega_{13}=\Omega_{31}=\frac{\beta^2(3\alpha^2-2\beta^2)}{(\alpha^2+\beta^2)(\alpha^4-3\alpha^2\beta^2+6\beta^4)}$$
$$\Omega_{22}=\frac{\alpha^2+3\beta^2}{\alpha^4-3\alpha^2\beta^2+6\beta^4}.$$

However, it is impossible to compute the maximizer of $S(\delta)$ in closed form.

Following [2][3] we construct a closed-form estimator with exactly the same asymptotic behaviour. We first construct two-point estimators: For $j=-1,1$, put $R_j = \operatorname{Re}(J_Y(\omega_{n_T+j})/J_Y(\omega_{n_T}))$. Then, since the $V_j = V_T(\omega_j)$ are $O_P\left(T^{1/2}\right)$ and $E=O(T)$,

$$\begin{aligned}\frac{J_Y(\omega_{n_T+j})}{J_Y(\omega_{n_T})} &= \frac{Ed_j(\delta)+V_j}{Ed_0(\delta)\left(1+\frac{V_0}{Ed_0(\delta)}\right)} \\ &= \frac{1}{Ed_0(\delta)}\{Ed_j(\delta)+V_j\}\left\{1-\frac{V_0}{Ed_0(\delta)}\right\} \\ &= \frac{d_j(\delta)}{d_0(\delta)}+\frac{V_j}{Ed_0(\delta)}-\frac{d_j(\delta)V_0}{Ed_0^2(\delta)}+O_P\left(T^{-1}\right).\end{aligned}$$

Thus

$$R_j=\frac{d_j(\delta)}{d_0(\delta)}+\operatorname{Re}\left(\frac{V_j}{Ed_0(\delta)}-\frac{d_jV_0}{Ed_0^2(\delta)}\right)+O_P\left(T^{-1}\right),$$

and to first order $R_j \sim \frac{d_j(\delta)}{d_0(\delta)}$. Now

$$\frac{d_j(\delta)}{d_0(\delta)}=\frac{\alpha\frac{\delta}{\delta-j}-\beta\frac{\delta}{\delta+1-j}-\beta\frac{\delta}{\delta-1-j}}{\alpha-\beta\frac{\delta}{\delta+1}-\beta\frac{\delta}{\delta-1}},$$

and so

$$\frac{d_1(\delta)}{d_0(\delta)} = \frac{\alpha\frac{\delta}{\delta-1} - \beta - \beta\frac{\delta}{\delta-2}}{\alpha - \beta\frac{\delta}{\delta+1} - \beta\frac{\delta}{\delta-1}} = \frac{\delta+1}{\delta-2} \times \frac{(\alpha-2\beta)\delta^2 - 2(\alpha-2\beta)\delta - 2\beta}{(\alpha-2\beta)\delta^2-\alpha}.$$

Thus $R_1 = d_1(\delta)/d_0(\delta)$ is cubic in δ, unless $\alpha = 2\beta$, i.e. unless the taper is the Hanning taper, which we assume from now on. We then have

$$\frac{d_1(\delta)}{d_0(\delta)} = \frac{(\delta-1)\delta(\delta+1)}{(\delta-1-j)(\delta-j)(\delta+1-j)} = \begin{cases} \frac{\delta-1}{\delta+2} & ; \quad j=-1 \\ \frac{\delta+1}{\delta-2} & ; \quad j=1. \end{cases}$$

We can therefore obtain two estimators of δ by solving for $j = -1, 1$, $R_j = d_j(\delta)/d_0(\delta)$. The solutions are

$$\delta = \begin{cases} \widehat{\delta}_{-1} = \frac{2R_{-1}+1}{1-R_{-1}} \\ \widehat{\delta}_1 = \frac{2R_1+1}{R_1-1}. \end{cases}$$

Letting $U_j = U_T(\omega_{n_T+j})$, we have

$$\frac{V_{-1}}{Ed_0(\delta)} - \frac{d_{-1}(\delta)V_0}{Ed_0^2(\delta)} = \frac{1}{Ed_0(\delta)}\left\{2U_{-1} - U_{-2} - U_0 - \frac{\delta-1}{\delta+2}(2U_0 - U_{-1} - U_1)\right\},$$

$$\frac{V_1}{Ed_0(\delta)} - \frac{d_1(\delta)V_0}{Ed_0^2(\delta)} = \frac{1}{Ed_0(\delta)}\left\{2U_1 - U_0 - U_2 - \frac{\delta+1}{\delta-2}(2U_0 - U_{-1} - U_1)\right\}.$$

Without loss of generality replace E by $|E|$ in the above. The asymptotic distribution of the real parts of the above may be shown to be normal with mean 0, as can that of

$$T^{1/2}\left(\widehat{\Delta} - \begin{bmatrix} \delta & \delta \end{bmatrix}'\right) = T^{1/2}\begin{bmatrix} \widehat{\delta}_{-1} - \delta & \widehat{\delta}_1 - \delta \end{bmatrix}'$$

If we choose as estimator of δ

$$\widehat{\delta} = \begin{cases} \widehat{\delta}_{-1} & ; \quad \widehat{\delta}_{-1}, \widehat{\delta}_1 > 0 \\ \widehat{\delta}_1 & ; \quad \text{otherwise,} \end{cases} \tag{8}$$

it can then be shown that $T^{1/2}\left(\widehat{\delta} - \delta\right)$ is asymptotically normal with mean 0 and variance

$$\frac{48\pi f(\omega_0)}{\rho^2(4\pi^2)} \times \frac{\pi^4\delta^2(\delta^2-1)^2}{12\sin^2(\pi\delta)} \times \frac{(2-|\delta|)^2(20\delta^2 - 20|\delta| + 14)}{9}.$$

In the same way as in [3], we can construct an "optimal" combination of $\widehat{\delta}_{-1}$ and $\widehat{\delta}_1$ which has the same asymptotics as the minimiser of $S(\delta)$. One such estimator is

$$\widehat{\delta} = \frac{\widehat{\delta}_{-1} + \widehat{\delta}_1}{2} + \kappa\left(\widehat{\delta}_{-1}^2\right) - \kappa\left(\widehat{\delta}_1^2\right), \tag{9}$$

where

$$\kappa(x) = -\frac{5}{14}\log\left(35x^2 + 120x + 32\right) + \frac{\sqrt{155}}{140}\log\left(\frac{7x + 12 - 4\sqrt{\frac{31}{5}}}{7x + 12 + 4\sqrt{\frac{31}{5}}}\right)$$

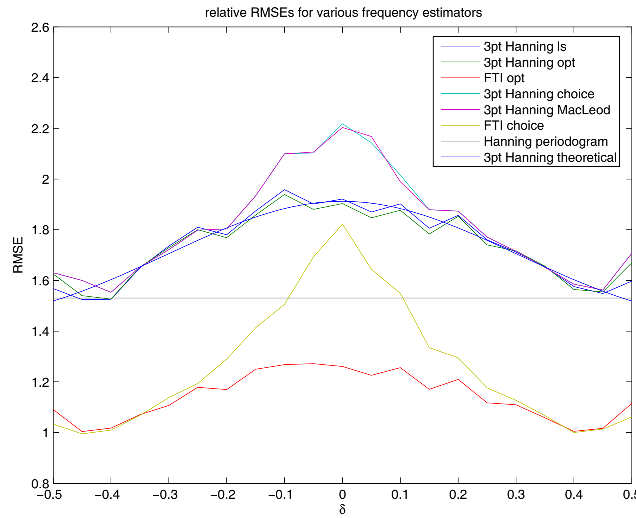

Fig. 1. Simulation results

is found using a simple integration argument.

Fig.1 displays the results of simulation experiments with various estimators. The curves plotted are root mean square errors relative to the asymptotic standard deviation of the ordinary periodogram maximizer (i.e. the square root of the Cramér-Rao lower bound for the case of Gaussian noise). For each $\delta \in (-0.5, 0.5)$ with 0.05 spacings, each estimate was computed for 1000 samples of size $T = 11025$. The signal-to-noise ratio used was -10dB. The highest to lowest values in the figure at $\delta = 0$ correspond to the estimate given by (8) (3pt Hanning choice), the analogous MacLeod [4] estimate (3pt Hanning MacLeod), the minimiser of $S(\delta)$ given by (7) (3pt Hanning ls), the theoretical asymptotic RMSE of the estimator given by (9) (3pt Hanning theoretical), the estimate given by (9) (3pt Hanning opt), the estimate given in [2] (FTI choice), the theoretical asymptotic RMSE for the maximizer of the periodogram of the Hanning-tapered periodogram (Hanning periodogram) and the estimate given in [3] (FTI opt). The results are as expected.

4. REFERENCES

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