ADAPTIVE-GAIN TRACKING FILTERS BASED ON MINIMIZATION OF THE INNOVATION VARIANCE

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ABSTRACT

A kinematic tracking filter is considered in the context of gain adaptation problem. The study suggests a simple adaptive-gain tracker based on minimization of the innovation variance. This is shown to provide the optimal Kalman gain. Accordingly, the innovation-based adaptive Kalman-like filter is constructed. The adaptive scheme is associated with a recursive MA-parameter estimator. With proper links for the optimal gain-vector components, the multiple-parameter adaptive filter reduces to a constrained single-parameter version. The simulation study justifies the filter performance for a wide range of conditions.

1. INTRODUCTION

The tracking (kinematic, polynomial, position-velocity, etc) filter remains in the focus of interest over recent decades. The 2nd and higher order trackers are commonly presented in the Kalman-filter form. The Kalman scheme is usually replaced by a steady-state, e.g., *α-β* or *α-β-γ* tracker [1-4] with a focus on the gain optimality w.r.t. the so-called tracking (maneuvering) index. In the practice, however, the uncertain or non-stationary conditions require the tracker adaptability.

Common adaptive versions of the tracking filter rely on such methods [4-6] as the multiple-model and interacting multiple-model filtering, covariance matching technique, residual 'whitening' and other, rather sophisticated schemes.

The present work suggests, by contrast, a simple adaptive-gain filter exploiting the standard parameter estimation technique. The suggested approach is to incorporate the tracker into a canonical adaptation framework, while the tracker gain is treated as an adaptation parameter.

Keeping in mind the 'whitening' property of the innovation, one may construct the adaptation scheme by minimizing the innovation variance. As shown in the paper, under given conditions the innovation-minimum-variance criterion is equivalent to the conventional Kalman-filter optimality criterion.

For the considered models of tracking filter, there are two available adaptation schemes. The first scheme identifies the (previously differenced) observation signal by a standard moving-averaged (MA) model and then translates the obtained parameters into the desired gain. The second method treats the gain components explicitly with the help of corresponding sensitivity functions.

As one applies the particular *α-β* and *α-β-γ* tracking filters, the auxiliary constraint linking optimal *α* and *β* (or *α*, *β* and *γ*) respectively) terms may be further used to reduce the space dimension of the adaptive filter parameters. This modification results in a single-parameter (e.g., *α*) constrained adaptive scheme appropriate for the two-state (or three-state, respectively) kinematic model.

2. FILTER BASICS

Let's consider a canonical discrete-time kinematic model

$$\mathbf{x}_{i+1} = \mathbf{F}\mathbf{x}_i + \mathbf{g}w_i \qquad y_i = \mathbf{h}\mathbf{x}_i + v_i \tag{1}$$

where i – discrete time, $\mathbf{x}_i$ – $N\times1$ state-vector of the kinematic parameters, i.e., position and its derivatives, y_i – observation, $\mathbf{F}$ – $N\times N$ transition matrix, $\mathbf{h}$ – $1\times N$ measurement matrix, $\mathbf{g}$ – control vector, w and v are mutually uncorrelated process and measurement noises, respectively, with variances $Q=\sigma_w^2$ and $R=\sigma_v^2$. The tracking filter follows the Kalman-like state-equation

$$\hat{\mathbf{x}}_{i+1} = \mathbf{F}\hat{\mathbf{x}}_i + \mathbf{k}_{i+1}e_{i+1}, \quad e_{i+1} = y_{i+1} - \mathbf{h}\mathbf{F}\hat{\mathbf{x}}_i \tag{2}$$

where the superscript '^' marks the estimator (for corresponding state-vector), e_i denotes innovation, and $\mathbf{k}_i$ - gain-vector. The asymptotic gain $\mathbf{k}=\lim(\mathbf{k}_i)$ ($i\to\infty$) is defined as [1]

$$\mathbf{k} = \mathbf{P}^{(-)}\mathbf{h}^T \big/ S = \mathbf{P}^{(+)}\mathbf{h}^T \big/ R \tag{3}$$

with the covariance update equations

$$\mathbf{P}^{(+)} = \mathbf{P}^{(-)} - \mathbf{k}\mathbf{h}\mathbf{P}^{(-)} \tag{4}$$

$$\mathbf{P}^{(-)} = \mathbf{F}\mathbf{P}^{(+)}\mathbf{F}^T + \mathbf{g}Q\mathbf{g}^T \tag{5}$$

where $\mathbf{P}^{(+)}=\{p^{(+)}_{i,j}\}$ and $\mathbf{P}^{(-)}=\{p^{(-)}_{i,j}\}$, $i,j=1,\ldots,N$ are the estimation and prediction covariance matrices, respectively,

$$S = E\left\langle \left(y_{i+1} - \mathbf{h}\mathbf{F}\hat{\mathbf{x}}_i\right)^2 \right\rangle = \mathbf{h}\mathbf{P}^{(-)}\mathbf{h}^T + R \tag{6}$$

is the innovation variance, and E – expectation sign.

There are particular recipes to identify the tracking filter gain $\mathbf{k}$. Thus the so-called *α-β* filter associates with the model

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ \dot{x}_i \end{bmatrix}, \mathbf{F} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \mathbf{g} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \tag{7}$$

where x_i and its first derivative are the position and velocity states, and T is the sampling period. The filter gain is specified as a vector with two normalized terms α and β, i.e.,

$$\mathbf{k} = \left(\alpha \quad \beta/T\right)^T \tag{8}$$

In the classical work [2], α and β are tied as

$$\beta = \alpha^2 \big/ (2-\alpha) \tag{9}$$

Later, Kalata [3] has introduced a generalized (tracking) index

$$\Lambda = \sigma_w T^2 / \sigma_n \tag{10}$$

Thus, given Λ, α and β can be derived as [4]:

$$\alpha = -0.125\left(\Lambda^2 + 8\Lambda - (\Lambda+4)\sqrt{\Lambda^2 + 8\Lambda}\right)$$
$$\beta = 0.25\left(\Lambda^2 + 4\Lambda - \Lambda\sqrt{\Lambda^2 + 8\Lambda}\right) \tag{11}$$

Given α, we can find β from the optimal [instead of (9)] link

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{12}$$

Another common tracker, α-β-γ filter, relies on the model

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{bmatrix}, \mathbf{F} = \begin{pmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{g} = \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \quad (13)$$

where the 3-state vector $\mathbf{x}_i$ comprises, in addition, the second derivative, i.e., acceleration and, respectively, the gain-vector is

$$\mathbf{k} = \left(\alpha \quad \beta/T \quad \gamma/2T^2\right)^T \quad (14)$$

where γ is an auxiliary term. The link (12) between optimal α and β holds, while the optimal γ follows from the formula [3]

$$\gamma = \beta^2/\alpha \quad (15)$$

The minimum variance of e_i, irrespective of the filter order, is

$$S_o = \min_{\mathbf{k}} S = \sigma_v^2/(1-\alpha) \quad (16)$$

As Λ is unknown, $\mathbf{k}$ should be determined adaptively. The suggested approach is to incorporate the above trackers into a canonical prediction error based adaptation framework [7], while the gain $\mathbf{k}$ is treated as an adaptation parameter.

3. GAIN OPTIMALITY

In applying the adaptation technique to a Kalman-like steady-state filter, it is convenient to replace the prediction error by innovation. It is known [8] that if the transition matrix $\mathbf{F}$ and observation matrix $\mathbf{h}$ are available, the value of $\mathbf{k}$ which whitens the innovation process (and minimizes the innovation variance) is the optimal Kalman gain.

Note that the Kalman gain (3) obeys the optimality condition

$$\frac{\partial Tr\left[\mathbf{P}^{(+)}\right]}{\partial \mathbf{k}} = 0 \quad (17)$$

where '*Tr*' is short for the matrix trace,

$$Tr\left[\mathbf{P}^{(+)}\right] = \sum_{1}^{k} p_{i,i}^{(+)} \quad (18)$$

On the second hand, minimizing S [defined in (6)] gives

$$\frac{\partial S}{\partial \mathbf{k}} = \frac{\partial p_{1,1}^{(-)}}{\partial \mathbf{k}} = 0 \quad (19)$$

In accordance to the theory, the Kalman gain should be optimal for the underlying (19) combination of the filter states. Below we will demonstrate the equivalence of (17) and (19) for our particular constructions.

3.1 α-β Tracking Filter.

The minimized prediction-error variance may be expanded as

$$p_{1,1}^{(-)} = p_{1,1}^{(+)} + p_{2,2}^{(+)}T^2 + 2p_{1,2}^{(+)}T + QT^4/4 \quad (20)$$

Assume that $\mathbf{P}^{(-)}$ is a steady-state covariance associated with the optimal α-β filter. Accordingly, $\mathbf{P}^{(+)}$ may be derived as

$$\begin{aligned} p_{1,1}^{(+)} &= (1-\alpha)^2 p_{1,1}^{(-)} + \alpha^2 R \\ p_{2,2}^{(+)} &= (\beta/T)^2\left(p_{1,1}^{(-)} + R\right) - 2(\beta/T)p_{1,2}^{(-)} + p_{2,2}^{(-)} \\ p_{1,2}^{(+)} &= \left[p_{1,2}^{(-)} - (\beta/T)p_{1,1}^{(-)}\right](1-\alpha) + R\alpha(\beta/T) \end{aligned} \quad (21)$$

The non-zero derivatives w.r.t. α and β (up to a scale factor) are

$$\begin{aligned} \frac{\partial p_{1,1}^{(+)}}{\partial \alpha} &= \frac{\partial p_{1,2}^{(+)}}{\partial \beta} \propto -p_{1,1}^{(-)}(1-\alpha) + R\alpha \\ \frac{\partial p_{2,2}^{(+)}}{\partial \beta} &= \frac{\partial p_{1,2}^{(+)}}{\partial \alpha} \propto -\left[p_{1,2}^{(-)} - (\beta/T)p_{1,1}^{(-)}\right] + (\beta/T)R \end{aligned} \quad (22)$$

Equating the first equation (22) to zero gives the Kalman-like

$$\alpha = p_{1,1}^{(-)}/\left(p_{1,1}^{(-)} + R\right) \quad (23)$$

while the second equation (22) gives, in turn, the Kalman-like

$$\beta/T = p_{2,1}^{(-)}/\left(p_{1,1}^{(-)} + R\right) \quad (24)$$

The non-diagonal cross-covariance element [third line (21)] combines the Kalman optimality conditions both for α and β.

The loss function (20) is minimized with the same α and β that minimize the loss function (18).

3.2. α-β-γ Tracking Filter.

The prediction-error variance is now expanded as

$$\begin{aligned} p_{1,1}^{(-)} = {} & p_{1,1}^{(+)} + 2p_{1,2}^{(+)}T + p_{1,3}^{(+)}T^2 + p_{2,2}^{(+)}T^2 + p_{2,3}^{(+)}T^3 \\ & + p_{3,3}^{(+)}T^4/4 + QT^4/4 \end{aligned} \quad (25)$$

As above, assuming $\mathbf{P}^{(-)}$ is known we can find $\mathbf{P}^{(+)}$. Noteworthy that the 3×3 matrix $\mathbf{P}^{(+)}$ of the 3-state α-β-γ filter comprises the 2×2 matrix (21) of the 2-state α-β filter as a diagonal block. In addition to (21), the resultant matrix $\mathbf{P}^{(+)}$ has new terms

$$\begin{aligned} p_{3,3}^{(+)} &= (\gamma/2T)^2\left(p_{1,1}^{(-)} + R\right) - 2\left(\gamma/2T^2\right)p_{1,3}^{(-)} + p_{3,3}^{(-)} \\ p_{3,2}^{(+)} &= \left[2p_{1,3}^{(-)} - \left(\gamma/2T^2\right)p_{1,1}^{(-)}\right](\beta/T) \\ &\quad + R\gamma\left(\beta/2T^3\right) + p_{2,3}^{(-)} + QT^4/4 \end{aligned} \quad (26)$$

The derivatives w.r.t. α, β and γ, up to a scale factor, are

$$\begin{aligned} \frac{\partial p_{1,1}^{(+)}}{\partial \alpha} &= \frac{\partial p_{2,1}^{(+)}}{\partial \beta} = \frac{\partial p_{3,1}^{(+)}}{\partial \gamma} \propto -p_{1,1}^{(-)}(1-\alpha) + R\alpha \\ \frac{\partial p_{2,2}^{(+)}}{\partial \beta} &= \frac{\partial p_{2,1}^{(+)}}{\partial \alpha} = \frac{\partial p_{3,2}^{(+)}}{\partial \gamma} \propto -\left[p_{2,1}^{(-)} - (\beta/T)p_{1,1}^{(-)}\right] + R(\beta/T) \\ \frac{\partial p_{3,3}^{(+)}}{\partial \gamma} &= \frac{\partial p_{3,2}^{(+)}}{\partial \beta} \propto -\left[p_{3,1}^{(-)} - \left(\gamma/2T^2\right)p_{1,1}^{(-)}\right] + R\left(\gamma/2T^2\right) \end{aligned} \quad (27)$$

Equating first two relations (27) to zero repeats (23) and (24), while the third relation (27) gives the Kalman-like

$$\gamma/2T^2 = p_{3,1}^{(-)}/\left(p_{1,1}^{(-)} + R\right) \quad (28)$$

Therefore minimization of the loss function (25) provides the Kalman gains (23), (24) and (28).

4. ADAPTIVE TRACKER

Considering the adaptation problem, we will specify the transfer function (t.f.) between the observation and innovation,

$$W_{ye} = 1 - q^{-1}\mathbf{hF}\left[\mathbf{I} - q^{-1}(\mathbf{I} - \mathbf{kh})\mathbf{F}\right]^{-1}\mathbf{k} \quad (29)$$

where q^{-1} denotes the one-step delay operator. From (29) follows

$$W_{ye} = \Delta^{(k)}/D_k \quad (30)$$

k=2,3, where $\Delta^{(k)}$ denotes a k-order difference, while

$$D_2 = 1 + (\alpha + \beta - 2)q^{-1} + (1-\alpha)q^{-2} \quad (31)$$

$$\begin{aligned} D_3 = {} & 1 + (\alpha + \beta + \gamma - 3)q^{-1} + (3 - 2\alpha - \beta + \gamma)q^{-2} \\ & + (\alpha - 1)q^{-3} \end{aligned} \quad (32)$$

Eq. (30) defines the innovation sequence as governed by the observation y_i autoregressive (AR) process coupled with the kth difference. In turn, the observation follows from the inverse t.f.

$$W_{ey} = D_k/\Delta^{(k)} \quad (33)$$

as a properly integrated MA process driven by the white (innovation) noise e_i.

The parameters of the t.f. (33) and components of **k** are linearly tied [see (31)-(32)]. Therefore the gain-adaptation problem reduces to an identification of the MA-model of the kth difference of y_i. Summarizing the method (Method 1), we should find a kth difference of the observation y_i, then estimate parameters of the (k-order) MA model and, finally, map these parameters into **k**. The core step – MA-parameter estimation can be readily performed by a standard identification procedure.

Alternatively, with the Method 2, one may apply the Kalman state-equation (2), while the gain is updated using the explicit derivatives of the innovation w.r.t. **k**. The Method 2 relies on the following below derivatives (sensitivity functions).

4.1. *α–β* Adaptive Tracking Filter.

The derivatives of e_i w.r.t. the (slowly varying) α and β are

$$\frac{\partial e_i}{\partial \alpha} = -\frac{\Delta^{(2)} D_2'}{D_2^2} y_i = -q^{-1}\Delta\tilde{e}_i \qquad (34)$$
$$\frac{\partial e_i}{\partial \beta} = -q^{-1}\tilde{e}_i$$

where,

$$\tilde{e}_i = e_i / D_k = e_i + \breve{D}_k \tilde{e}_{i-1} \qquad (35)$$

is the pre-filtered error, $\breve{D}_k(q)=q[1\text{-}D_k(q)]$ (presently with k=2). Eqs. (34)-(35) define a sensitivity function with two adaptation parameters, α and β. A particular adaptation mechanism (RLS, LMS) may be selected from those available in the literature [7].

The method assuming direct update of **k** allows some reasonable modifications. Since the innovation-minimum-variance gain fits the Kalman gain, we can reduce the number of adjusted parameters by constraining α and β with (9) [or (12)]. Considering β as a function of α changes the first line (34) to

$$\frac{\partial e_i}{\partial \alpha} = -(1+\beta')\tilde{e}_{i-1} + \tilde{e}_{i-2} = -q^{-1}\Delta\tilde{e}_i - \beta'\tilde{e}_{i-1} \qquad (36)$$

where, using (9), we can find

$$\beta' = \frac{\partial \beta}{\partial \alpha} = \left[2\alpha(2-\alpha)+\alpha^2\right] / (2-\alpha)^2 = 2\alpha/(2-\alpha) \qquad (37)$$

or, using (12),

$$\beta' = -2 + 2/\sqrt{1-\alpha} \qquad (38)$$

Applying either gradient (37) or (38) (together with a proper expression for β) results in a single-parameter adaptive scheme. A closed-form expression for the gradient can be expanded using a particular function for β. Substituing (37) into (36) results in

$$\frac{\partial e_i}{\partial \alpha} = -q^{-1}\left[4(\alpha-2)^{-2} - q^{-1}\right]\tilde{e}_i \qquad (39)$$

To give the constrained adaptive filter even a simpler 'truncated' form, we can abandon the extra term in (36) and update α due to (34), while β follows from (9) [or (12)], as usual.

4.2. *α–β-γ* Adaptive Tracking Filter.

The derivatives of e_i w.r.t. (slowly varying) α, β and γ are

$$\frac{\partial e_i}{\partial \alpha} = -\frac{\Delta^{(3)} D_3'}{D_3^2} y_i = -q^{-1}\Delta^{(2)}\tilde{e}_i$$
$$\frac{\partial e_i}{\partial \beta} = -\frac{\Delta^{(2)} D_3'}{D_3^2} y_i = -q^{-1}\Delta\tilde{e}_i \qquad (40)$$
$$\frac{\partial e_i}{\partial \gamma} = -\frac{1}{4}\left(q^{-1}+q^{-2}\right)\tilde{e}_i$$

where the pre-filtered innovation term obeys (35) (with k=3).

Analogously, we may rearrange the three-parameter filter into a single-parameter. Viewing β and γ as functions of α gives

$$\frac{\partial e_i}{\partial \alpha} = -(1+\beta' + \frac{1}{4}\gamma')\tilde{e}_{i-1} - (\frac{1}{4}\gamma' - \beta' - 2)\tilde{e}_{i-2} - \tilde{e}_{i-3} \qquad (41)$$

with the following from (15)

$$\gamma' = \partial\gamma/\partial\alpha = \beta(2\alpha\beta' - \beta)/\alpha^2 \qquad (42)$$

applied in conjunction with (37) [or (38)].

For the truncated form ($\beta'=\gamma'=0$) of the single-parameter adaptive filter, the Eq. (41) is reduced to the first equation (40).

The above variants of the adaptive tracking filter were implemented and tested in different simulation scenarios.

5. SIMULATION

The simulation study justifies the optimality of the adaptive gain. It also compares the single-parameter (full and truncated) version of the adaptive filter to its multi-parameter origin.

The first experiment illustrates the steady-state performance of the adaptive trackers for different tracking indices (Figs. 1-2).

In each run, y_i is generated due to the model (with k=2 or 3)

$$y_i = \left[0.5T^2\left(1+q^{-1}\right)/\left(1-q^{-1}\right)^k\right]w_i + v_i \qquad (43)$$

excited by the observation and plant noises, with T=1 and particular Q and R. The output signal is adopted, in turn, by a multiple-parameter adaptive tracker (with Method 1 or 2) and by its single-parameter variant (full or reduced), all with the RLS.

As the filter converges and the innovation approaches the white-noise sequence (what occurs very rapidly), the gain is averaged over the remaining time and the resulting statistics are collected. This procedure is repeated and the final gain is averaged over 10 runs. These trials are performed with different Λ changing from 10^{-4} to 10^2 (with Q=1 assumed constant, while R varied properly).

The model (43) with k=2 was used to generate the signal with the 1st-order trend. Fig. 1 compares the adaptive α and β obtained by different algorithms versus the theoretical α (solid curve) and β (dash), respectively. First, Fig. 1 depicts the adaptive α (diamond) and β (square) related to the two-parameter tracker implemented either by the Method 1 or 2 (both gave similar results). Next, Fig.1 shows the adaptive α (circle) associated with the single-parameter tracker and, equivalently, its truncated version. In mean, the adaptive α and β are very close to theory.

Analogously, the model (43) with k=3 was used to generate the signal with the 2nd-order trend. Fig. 2 illustrates the performance of the three-state adaptive tracking filter and its variants.

As it was observed from the trials, the single-parameter version of the adaptive filter demonstrates a lower gain-adjustment noise than its multiple-parameter counterpart. The monitoring was necessary only for the single-parameter filter utilizing the constraint (12) (which requires $\alpha < 1$). Other filters hold stability even if α exceeds 1.

The truncated single-parameter adaptive *α-β* (or *α-β-γ*) filter does not deviate considerably from its full-gradient form.

The second experiment is designed to illustrate the tracking capability of the constrained *α-β* and *α-β-γ* filters (Figs. 3 and 4, respectively). We assume that Λ jumps after each $2\cdot10^4$ samples from $\log_{10}(\Lambda)$=–3 to 3, by 1, and then returns back to –3.

As the maneuvering index varies, the single-parameter adaptive tracker properly adjusts α and approaches the expected optimal gain. We may note a short transient and gain fluctuations near the optimum. The performed simulations showed that the adaptive gain always approaches the proper value irrespective of the order and size of jumps. This assures us in the adequate ability of the adaptive-gain kinematic filter to track the non-stationary dynamic processes in a wide range of conditions.

The multiple-parameter tracker also successfully copes with the latter scenario however with a larger gain-adjustment noise.

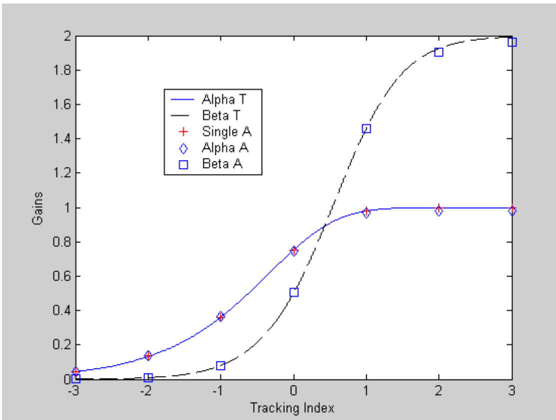

Fig. 1. Tracker gains: 'T'-theory, 'A' - adaptive filters, 'Single' - single-parameter tracker.

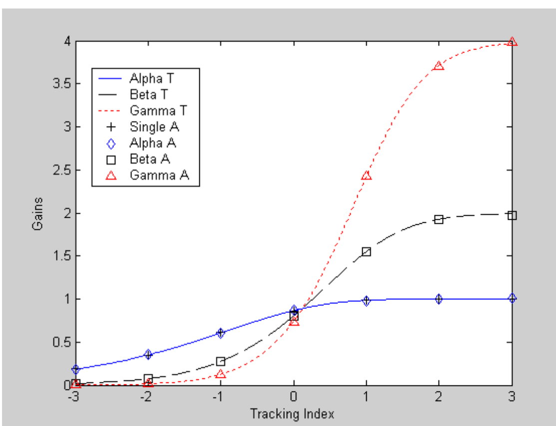

Fig. 2. Tracker gains: 'T'-theory, 'A' - adaptive filters, 'Single' - single-parameter tracker.

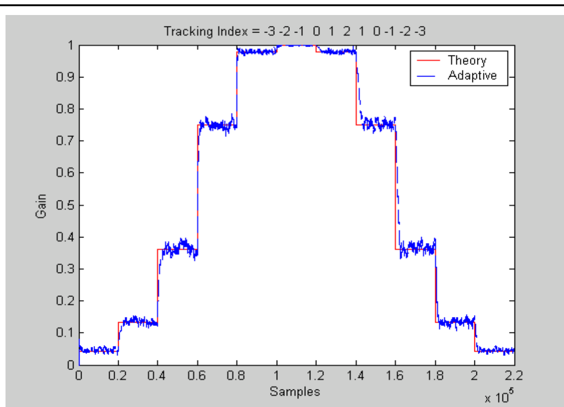

Fig. 3. Adaptive α (dash) vs optimal α (solid). Λ varies each $2 \cdot 10^4$ samples, from 10^{-3} to 10^3, by 10, and back to 10^{-3}.

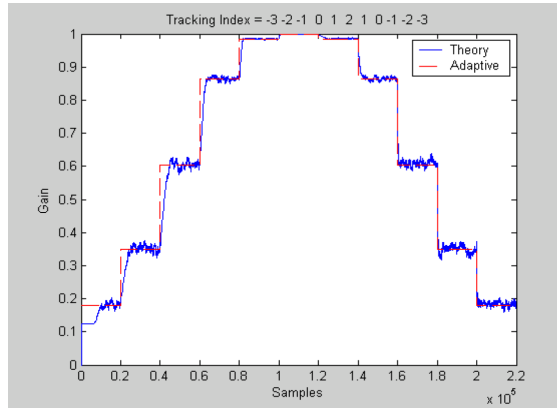

Fig. 4. Adaptive α (dash) vs optimal α (solid). Λ varies each $2 \cdot 10^4$ samples, from 10^{-3} to 10^3, by 10, and back to 10^{-3}.

6. CONCLUDING REMARKS

In the present study, the steady-gain *α-β* and *α-β-γ* tracking filters are given an adaptive-gain form. For the steady-gain tracker, the innovation-minimum-variance criterion is shown to provide the optimal Kalman gain. Accordingly, the innovation-based adaptive Kalman-like tracking filter is constructed.

As shown, the adaptive-gain trackers reduce to a canonical recursive estimator of the MA-filter parameters. Thus, the suggested three-stage Method 1 combines a difference of the raw observation signal, estimation of the MA-model parameters, and, finally, their linear transform into the filter gain-vector. An alternative one-stage Method 2 updates the gain directly using the innovation sequence produced by the Kalman filter.

The tracking filter form allows some reasonable modifications. Thus, constraining the gain components, the multiple-parameter filter is reduced to a single-parameter construction.

The developed adaptive trackers were tested experimentally for a wide range of conditions. In all runs, the gain approaches the optimal value predicted by theory. Both Methods 1 and 2 (the latter in the multiple-parameter version) show same results.

The single-parameter version indicates the lower gain-adjustment noise compared to the multiple-parameter case. A comprehensive analysis of the constrained single-parameter, particularly the non-linear filter is the issue of further studies.

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