REDUCED SIGMA POINT FILTERING FOR PARTIALLY LINEAR MODELS

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ABSTRACT

A method for performing unscented Kalman filtering with a reduced number of sigma points is proposed. The procedure is applicable when either the process or measurement equations are partially linear in the sense that only a subset of the elements of the state vector undergo a nonlinear transformation. It is shown that for such models second-order accuracy in the moments required for the unscented Kalman filter recursion can be obtained using a number of sigma points determined by the number of nonlinearly transformed elements rather than the dimension of the state vector. A procedure for computing the sigma points is developed. An application of the proposed method to smoothed target state estimation from bearings measurements is presented.

1. INTRODUCTION

We consider a stochastic dynamic system with the state $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$, $k = 0, 1, \ldots$ a Markov process evolving according to

$$\boldsymbol{x}_k = \boldsymbol{f}(\boldsymbol{x}_{k-1}) + \boldsymbol{v}_k, \quad k = 1, 2, \ldots, \tag{1}$$

where the process noise $\{\boldsymbol{v}_k\}$ is zero-mean and satisfies $\mathsf{cov}(\boldsymbol{v}_k, \boldsymbol{v}_l) = \boldsymbol{Q}_k \delta_{k-l}$ for all k, l. The initial state $\boldsymbol{x}_0 \sim \pi_0$. Indirect observations $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$, $k = 1, 2, \ldots$, of the state are generated according to

$$\boldsymbol{y}_k = \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{w}_k, \quad k = 1, 2, \ldots, \tag{2}$$

where the measurement noise $\{\boldsymbol{w}_k\}$ is zero-mean and satisfies $\mathsf{cov}(\boldsymbol{w}_k, \boldsymbol{w}_l) = \boldsymbol{R}_k \delta_{k-l}$ and $\mathsf{cov}(\boldsymbol{w}_k, \boldsymbol{v}_l) = \boldsymbol{0}_{n_y, n_x}$ for all k, l with $\boldsymbol{0}_{i,j}$ a $i \times j$ matrix of zeros. The problem of interest is to estimate the state from the observed measurements. The minimum mean square estimator of the the state at time k from measurements up to time k is the posterior expectation $\mathsf{E}(\boldsymbol{x}_k | \boldsymbol{y}_{1:k})$ where $\boldsymbol{y}_{a:b}$ is the collection of measurements from time a to time b.

For general nonlinear/non-Gaussian models the posterior expectation cannot be computed. In such cases the Kalman filter (KF) provides a computationally efficient estimate with the useful property of being the linear minimum variance estimate [2, 7]. Well-known approximations to the KF include the extended KF (EKF) [4], obtained by linearisation, and the unscented KF (UKF) [5], obtained by using the unscented transformation (UT).

The UKF is particularly useful as it has about the same computational expense as the EKF but does not require the Jacobian matrix and is more accurate, significantly so in certain cases. The core of the UKF is the UT which approximates the moments of a nonlinearly transformed random variable by the sample moments of a set of sigma points passed through the nonlinear transformation. A common characteristic of the several UT variants is that the number of sigma points is an increasing function of the dimension of the random variable undergoing the transformation [6].

In many applications of practical interest the transformations $\boldsymbol{f}$ and $\boldsymbol{h}$ in (1) and (2) are only partially nonlinear in the sense that they are applied only to certain elements of the state vector. This is common, for instance, in target tracking problems where measurements depend only on the position elements of the state vector and are independent of the velocity and acceleration elements [1]. In this paper it will be shown that, for a class of nonlinear models we refer to as partially linear, the moments required for the KF recursion can be approximated with second-order accuracy using the UT with a number of sigma points determined by the dimension of a subset of the elements of the state vector. This permits a reduction in computational expense while ensuring accuracy is maintained. Although the implementation is quite different, the idea is similar to Rao-Blackwellisation in particle filtering wherein partial linearity is exploited by using a particle filter only for those elements of the state which are nonlinearly transformed [9]. Inference for the remaining elements is performed using a KF.

The paper is organised as follows. The UKF is reviewed in Section 2. The reduced sigma point transformation is presented in Section 3 and an application is given in Section 4. Conclusions are drawn in Section 5.

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2. THE UNSCENTED KALMAN FILTER

The linear minimum variance estimate of the state $\boldsymbol{x}_k$ is obtained using the KF [7]. The KF characterises the state estimate at time $k-1$ by a mean $\boldsymbol{x}_{k-1|k-1}$ and covariance matrix $\boldsymbol{P}_{k-1|k-1}$. For general nonlinear/non-Gaussian models, these moments can be interpreted as approximations of the true posterior mean and covariance matrix. According to the maximum entropy principle the posterior density is taken to be Gaussian [2] so that the posterior density at time $k-1$ is approximated as,

$$\hat{p}(\boldsymbol{x}_{k-1}|\boldsymbol{y}_{1:k-1}) = N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1}), \quad (3)$$

where $N(\boldsymbol{z}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the Gaussian pdf with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ evaluated at $\boldsymbol{z}$. The first step in the KF recursion is to compute the predicted mean and covariance matrix,

$$\hat{\boldsymbol{x}}_{k|k-1} = \mathsf{E}(\boldsymbol{f}(\boldsymbol{X}_{k-1})|\boldsymbol{y}_{1:k-1}), \quad (4)$$
$$\boldsymbol{P}_{k|k-1} = \mathsf{cov}(\boldsymbol{x}_k, \boldsymbol{x}_k|\boldsymbol{y}_{1:k-1}), \quad (5)$$

where expectations are taken with respect to the density (3). The predicted mean and covariance matrix are then corrected using the current measurement to give

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}), \quad (6)$$
$$\boldsymbol{P}_{k|k} = \boldsymbol{P}_{k|k-1} - \boldsymbol{K}_k\boldsymbol{\Psi}_k', \quad (7)$$

where

$$\hat{\boldsymbol{y}}_{k|k-1} = \mathsf{E}(\boldsymbol{h}(\boldsymbol{x}_k)|\boldsymbol{y}_{1:k-1}), \quad (8)$$
$$\boldsymbol{S}_k = \mathsf{cov}(\boldsymbol{y}_k, \boldsymbol{y}_k|\boldsymbol{y}_{1:k-1}), \quad (9)$$
$$\boldsymbol{\Psi}_k = \mathsf{cov}(\boldsymbol{x}_k, \boldsymbol{y}_k|\boldsymbol{y}_{1:k-1}), \quad (10)$$

with $\boldsymbol{K}_k = \boldsymbol{\Psi}_k\boldsymbol{S}_k^{-1}$ and expectations are with respect to

$$\hat{p}(\boldsymbol{x}_k|\boldsymbol{y}_{1:k-1}) = N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1}). \quad (11)$$

The conditional expectations (4), (5) and (8)-(10) cannot in general be computed exactly. The UT is a numerical technique for approximating these expectations.

2.1. The unscented transformation

We describe the UT as presented in [5]. Consider a random variable $\boldsymbol{r} \in \mathbb{R}^n$ with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ passed through a nonlinear transformation $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$. The UT approximates the moments of the random variable $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{r})$ by propagating through $\boldsymbol{g}$ a set of weighted sigma points selected to match the mean and covariance matrix of $\boldsymbol{r}$. For the covariance matrix $\boldsymbol{\Sigma}$, define the square root $\sqrt{\boldsymbol{\Sigma}}$ such that $\sqrt{\boldsymbol{\Sigma}}\left(\sqrt{\boldsymbol{\Sigma}}\right)' = \boldsymbol{\Sigma}$. The ith column of a matrix $\boldsymbol{\Sigma}$ is denoted as $\boldsymbol{\Sigma}_i$. The set of sigma points $\mathcal{R}^0, \ldots, \mathcal{R}^{2n}$ are then selected as

$$\mathcal{R}^i = \begin{cases} \boldsymbol{\mu}, & i = 0, \\ \boldsymbol{\mu} + \left(\sqrt{(n+\kappa)\boldsymbol{\Sigma}}\right)_i, & i = 1, \ldots, n, \\ \boldsymbol{\mu} - \left(\sqrt{(n+\kappa)\boldsymbol{\Sigma}}\right)_{i-n}, & i = n+1, \ldots, 2n, \end{cases} \quad (12)$$

where κ is a parameter to be selected. The sigma points are given the weights

$$w^i = \begin{cases} \kappa/(n+\kappa), & i = 0, \\ 1/\{2(n+\kappa)\}, & i = 1, \ldots, 2n. \end{cases} \quad (13)$$

The sample mean and covariance matrix of the sigma points are equal to $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively [5]. Each of the sigma points is passed through $\boldsymbol{g}$ to obtain $\mathcal{Z}^i = \boldsymbol{g}(\mathcal{R}^i)$, $i = 0, \ldots, 2n$. The mean and covariance matrix of $\boldsymbol{z}$ and the cross-covariance matrix $\mathsf{cov}(\boldsymbol{r}, \boldsymbol{z})$ can then be approximated using the sample moments of the sigma points $\mathcal{R}^0, \ldots, \mathcal{R}^{2n}$, $\mathcal{Z}^0, \ldots, \mathcal{Z}^{2n}$. The moment approximations are accurate up to second-order [5], i.e., the moments of the Taylor series expansion of $\boldsymbol{g}(\boldsymbol{r})$ taken about $\boldsymbol{\mu}$ agree with the approximations up to and including the second-order terms.

3. REDUCED SIGMA POINT TRANSFORMATION

In this section it is shown how, for certain transformations, the UT can be used to approximate the required moments with a reduced number of sigma points. In the following the ith element of a vector $\boldsymbol{v}$ is denoted as $v(i)$.

Let $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{r})$ where $\boldsymbol{r} \in \mathbb{R}^n$ has mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Assume the partitioning $\boldsymbol{r} = (\boldsymbol{a}', \boldsymbol{b}')'$ where $\boldsymbol{a} \in \mathbb{R}^{n_1}$ and $\boldsymbol{b} \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, such that

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \boldsymbol{g}(\boldsymbol{r}) = \begin{pmatrix} \boldsymbol{\gamma}(\boldsymbol{a}) \\ \boldsymbol{\Gamma}\boldsymbol{r} \end{pmatrix}. \quad (14)$$

Let $\boldsymbol{\nu} = \mathsf{E}(\boldsymbol{a})$, $\boldsymbol{\Omega} = \mathsf{cov}(\boldsymbol{a}, \boldsymbol{a})$ and $\boldsymbol{\Phi} = \mathsf{cov}(\boldsymbol{r}, \boldsymbol{a})$. We first show that matching $\boldsymbol{\mu}$ and $\boldsymbol{\Phi}$ is sufficient for second-order accuracy in the approximation of the required moments.

Proposition 1: For the random variable $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{r})$ of (14), the terms up to and including second-order in the Taylor series expansions of the moments $\mathsf{E}(\boldsymbol{z})$, $\mathsf{cov}(\boldsymbol{z}, \boldsymbol{c})$ and $\mathsf{cov}(\boldsymbol{r}, \boldsymbol{c})$ are independent of $\mathsf{cov}(\boldsymbol{b}, \boldsymbol{b})$.

Proof: For $\boldsymbol{g}$ as given in (14), the Taylor series expansion of $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{r})$ about $\boldsymbol{\mu}$ is

$$\boldsymbol{z} = \begin{pmatrix} \sum_{i=0}^{\infty} \mathbf{e}_i \\ \boldsymbol{\Gamma}\boldsymbol{r} \end{pmatrix}, \quad (15)$$

where

$$\mathbf{e}_i = \frac{1}{i!}\left(\sum_{j=1}^{n_1}(a(j) - \nu(j))\frac{\partial}{\partial a(j)}\right)^i \boldsymbol{\gamma}(\boldsymbol{a})|_{\boldsymbol{a}=\boldsymbol{\nu}}. \quad (16)$$

The expected value of (15) can be found as

$$\mathsf{E}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{\gamma}(\boldsymbol{\nu}) + \boldsymbol{\xi}(\boldsymbol{\nu}, \boldsymbol{\Omega}) + \sum_{i=3}^{\infty} \mathsf{E}(\mathbf{e}_i) \\ \boldsymbol{\Gamma}\boldsymbol{\mu} \end{pmatrix}, \quad (17)$$

where $\boldsymbol{\xi}(\boldsymbol{\nu}, \boldsymbol{\Omega})$ is a second-order term depending on the mean $\boldsymbol{\nu}$ and covariance matrix $\boldsymbol{\Omega}$ of $\boldsymbol{a}$. Since $\mathbf{e}_0 = \boldsymbol{\gamma}(\boldsymbol{\nu})$ and $\mathbf{e}_1 = \mathbf{J}_{\boldsymbol{\gamma}}(\boldsymbol{a} - \boldsymbol{\nu})$ where $\mathbf{J}_{\boldsymbol{\gamma}} = \nabla_{\boldsymbol{a}} \boldsymbol{\gamma}(\boldsymbol{a})'|_{\boldsymbol{a}=\boldsymbol{\nu}}$ and ∇ is the gradient operator, we have

$$\mathsf{cov}(\boldsymbol{z}, \boldsymbol{c}) = \begin{pmatrix} \mathbf{J}_{\boldsymbol{\gamma}} \boldsymbol{\Omega} \mathbf{J}_{\boldsymbol{\gamma}}' \\ \boldsymbol{\Gamma}\boldsymbol{\Phi}\mathbf{J}_{\boldsymbol{\gamma}}' \end{pmatrix} + \boldsymbol{H}, \quad (18)$$

where the $\boldsymbol{H}$ contains higher-order terms. Similarly,

$$\mathsf{cov}(\boldsymbol{r}, \boldsymbol{c}) = \boldsymbol{\Phi}\mathbf{J}_{\boldsymbol{\gamma}}' + \boldsymbol{L}, \quad (19)$$

where $\boldsymbol{L}$ contains higher-order terms. Eqs. (17), (18) and (19) show that the terms up to and including second-order of the required moments do not depend on the covariance matrix $\mathsf{cov}(\boldsymbol{b}, \boldsymbol{b})$.

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A direct corollary of Proposition 1 is that when the UT is used to approximate the second-order moments of $\boldsymbol{z}$ and $\boldsymbol{r}$ second-order accuracy can be achieved by selecting the sigma points to match the mean of $\boldsymbol{r}$ and the cross-covariance matrix between $\boldsymbol{r}$ and the partition $\boldsymbol{a}$. Note that the moments not accounted for in Proposition 1 can be computed exactly as $\mathsf{cov}(\boldsymbol{d}, \boldsymbol{d}) = \boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\Gamma}'$ and $\mathsf{cov}(\boldsymbol{r}, \boldsymbol{d}) = \boldsymbol{\Sigma}\boldsymbol{\Gamma}'$.

The sigma points are selected as

$$\mathcal{R}^i = \begin{pmatrix} \mathcal{A}^i \\ \mathcal{B}^i \end{pmatrix} = \boldsymbol{\mu} + \begin{pmatrix} \boldsymbol{\sigma}^i \\ \boldsymbol{\omega}^i \end{pmatrix}, \quad (20)$$

for $i = 0, \ldots, 2n_1 + 1$. The weights are selected as in (13) and the perturbations $\boldsymbol{\sigma}^0, \ldots, \boldsymbol{\sigma}^{2n_1}$ are the same as the perturbations in (12) with $\boldsymbol{\Sigma}$ set to $\boldsymbol{\Omega}$ and n set to n_1. The sigma points $\boldsymbol{\omega}^0, \ldots, \boldsymbol{\omega}^{2n_1}$ are selected to match the mean $\boldsymbol{\tau} = \mathsf{E}(\boldsymbol{b})$ and the cross-covariance matrix $\boldsymbol{\Delta} = \mathsf{cov}(\boldsymbol{a}, \boldsymbol{b})$. The mean can be matched by selecting a symmetric set of sigma points, i.e., $\boldsymbol{\omega}^0 = \mathbf{0}_{n_2,1}$ and $\boldsymbol{\omega}^i = -\boldsymbol{\omega}^{i+n_1}$, $i = 1, \ldots, n_1$. With this set of sigma points, the sample cross-covariance matrix between $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\sum_{i=0}^{2n_1} w^i \left(\boldsymbol{\nu} - \mathcal{A}^i\right)\left(\boldsymbol{\tau} - \mathcal{B}^i\right)' = \frac{1}{n_1 + \kappa} \sum_{i=1}^{n} \boldsymbol{\sigma}^i \boldsymbol{\omega}^{i\prime}. \quad (21)$$

Equating this sample covariance matrix to $\boldsymbol{\Delta}$ gives, after some working, the system of equations

$$\sqrt{\frac{\boldsymbol{\Omega}}{n_1 + \kappa}} \begin{pmatrix} \omega^1(j) \\ \vdots \\ \omega^{n_1}(j) \end{pmatrix} = \boldsymbol{\Delta}_j, \quad (22)$$

for $j = 1, \ldots, n_2$, where $\omega^i(j)$ is the jth element of $\boldsymbol{\omega}^i$.

4. EXAMPLE APPLICATION

In this section the reduced sigma point transformation of Section 3 is applied to the problem of smoothed state estimation of a moving target from bearings measurements. The nonlinearity in the measurement equation prevents computation of the optimal solution to this problem. Here smoothing over l lags is performed using the UKF to obtain filtered state estimates of an augmented state comprised, at time k, of the target states $\boldsymbol{x}_{k-l:k}$. Existing techniques for computationally efficient smoothing include an extended Kalman smoother (EKS) [8] and a refined technique, referred to here as EKS2 [3]. Comparisons between these methods, the unscented Kalman smoother (UKS) and the Cramér-Rao bound (CRB) will be performed.

The target state is $\boldsymbol{x}_k = (x_k, \dot{x}_k, y_k, \dot{y}_k)'$ where x_k is the target x-position and y_k is the target y-position. The evolution of the target state is modelled as in (1) with

$$\boldsymbol{f}(\boldsymbol{x}_{k-1}) = \left(\boldsymbol{I}_2 \otimes \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}\right) \boldsymbol{x}_{k-1}, \quad (23)$$

$$\boldsymbol{Q}_k = \boldsymbol{I}_2 \otimes q \begin{pmatrix} T^3/2 & T^2/2 \\ T^2/2 & T \end{pmatrix}, \quad (24)$$

where $\boldsymbol{I}_m$ is the $m \times m$ identity matrix, $\otimes$ is the Kronecker product, $T = 1$ is the sampling period and the process noise is Gaussian with intensity $q = 1 \times 10^{-3}$.

Bearings measurements are made by an observer with known position $\mathbf{s}_k = (\xi_k, \zeta_k)'$ at time kT. The measurement equation is given by (2) with

$$\boldsymbol{h}(\boldsymbol{x}_k) = \arctan\{(y_k - \zeta_k)/(x_k - \xi_k)\}, \quad (25)$$

$$\boldsymbol{R}_k = (2\pi/180)^2. \quad (26)$$

where the measurement noise is Gaussian. To ensure observability the sensor platform moves in a circular path of radius 35 centred at the origin of the plane. The target trajectory of 50 time steps is generated without process noise with initial target state $\boldsymbol{x}_0 = (30, 0, 840, -60)'$. The estimation algorithms are initialised with a Gaussian prior with mean given by the initial target state and covariance matrix $\boldsymbol{P}_0 = \mathrm{diag}(10, 1, 10, 1)$.

Assume that smoothing is to be performed with a fixed-lag $l \geq 1$. The augmented state approach formulates the smoothing problem as filtered state estimation of the augmented state vector $\boldsymbol{X}_k = (\boldsymbol{x}_k', \ldots, \boldsymbol{x}_{k-l}')'$. Let $\boldsymbol{X}_k^{(j)} = \boldsymbol{x}_{k-j+1}$, $j = 1, \ldots, l+1$ denote the jth partition of $\boldsymbol{X}_k$. The augmented state then evolves according to

$$\boldsymbol{X}_k = \boldsymbol{F}(\boldsymbol{X}_{k-1}) + \mathbf{V}_k, \quad (27)$$

where $\mathbf{V}_k = (\boldsymbol{v}_k', \mathbf{0}_{1,n_x l})'$ and

$$\boldsymbol{F}(\boldsymbol{X}_{k-1}) = \left(\boldsymbol{f}\left(\boldsymbol{X}_{k-1}^{(1)}\right)', \boldsymbol{X}_{k-1}^{(1)\prime}, \ldots, \boldsymbol{X}_{k-1}^{(l)\prime}\right)'. \quad (28)$$

Measurements are generated according to

$$\boldsymbol{y}_k = \boldsymbol{H}(\boldsymbol{X}_k) + \boldsymbol{w}_k, \tag{29}$$

where $\boldsymbol{H}(\boldsymbol{X}_k) = \boldsymbol{h}(\boldsymbol{X}_k^{(1)})$. The j-lag smoothed estimate of $\boldsymbol{x}_{k-j}$ is given by the partition $\hat{\boldsymbol{X}}_{k|k}^{(j+1)}$, $j = 0, \ldots, l$.

For the filtering problem described by (27) and (29) the UT is required only for the measurement update since the process equation (27) is linear and Gaussian. The procedure of Section 3 can be used to approximate the moments (8)-(10) required for the measurement update by noting that $\boldsymbol{H}$ in (29) is of the form (14) with $\boldsymbol{a} = (x_k, y_k)'$ and $\boldsymbol{b} = (\dot{x}_k, \dot{y}_k, \boldsymbol{X}_k^{(2)\prime}, \ldots, \boldsymbol{X}_k^{(l+1)\prime})'$. Since the dimension of $\boldsymbol{a}$ is $n_1 = 2$ only $2n_1 + 1 = 5$ sigma points are required. These can be found as shown in Section 3.

The performances of the various algorithms are measured by the RMS position error averaged over 1000 realisations and the first 30 time steps. Smoothing lags from zero to twenty are considered. Figure 1 shows the RMS position errors of the EKS, EKS2 and UKS plotted against smoothing lag. The CRB is also shown. As expected all algorithms improve as the smoothing lag increases. The best performance is achieved by the UKS with the EKS2 and EKS performing almost identically. The use of the reduced sigma point transformation in the UKS reduces computational expense by half compared to a straightforward application of the UT.

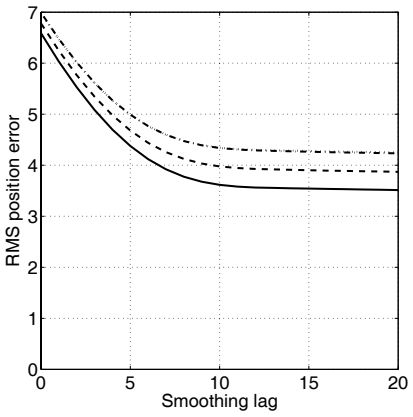

Figure 1: RMS position error plotted against smoothing lag for the EKS (dash-dot), EKS2 (dotted) and UKS (dashed). The CRB is shown as a solid line.

5. CONCLUSIONS

The use of the unscented Kalman filter (UKF) for state estimation in nonlinear/non-Gaussian models was considered. It was shown that the moments required in the UKF recursion can be accurately estimated with a reduced number of sigma points for models in which only a subset of the elements of the state vector are subject to a nonlinear transformation in the process or measurement equations. The proposed procedure was applied to smoothed state estimation in a nonlinear model.

There are various topics for future research. It would be of interest to determine the relationship of the proposed procedure to the various existing implementations of the unscented transformation. This would conceivably include analysis of accuracy of the method relative to the existing methods. A limitation of the proposed procedure is that the sigma points for the nonlinearly transformed elements must be selected according to the method originally proposed by Julier *et al.*, i.e., there must be $2n_1 + 1$ sigma points if there are n_1 nonlinearly transformed elements. It would be useful to lift this restriction.

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