TWO-DIMENSIONAL SUB-SAMPLE SHIFT ESTIMATION USING PLANE PHASE FITTING

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ABSTRACT

This paper investigates the problem of two-dimensional shift estimation between two sinusoids, proposing a method based on a least square plane fitting of the phases of two complex functions. The complex functions are defined using the cross-correlation and its Hilbert transforms. This estimation method is shown to be unbiased for long signals and high signal-to-noise ratios.

The case of truncated signals is considered and an iterative version of the estimator, giving more accurate results in these situations, is proposed.

1. INTRODUCTION

Two-dimensional shift estimation has its applications in different fields. In medical imaging, in particular in elastography, sub-pixel displacement between two ultrasound images needs to be estimated [1]. Most of the existing estimations are based on the maximum of the correlation function [2]. The shifts estimations are obtained as the shift lags that maximize the cross-correlation function.

As shown by Liebgott [3], the ultrasound images can follow the model presented in this paper, based on two shifted sinusoids (1). We propose an estimation using the plane phases of two complex functions, defined to provide linear phases in both directions.

In the literature, plane fitting is also used for frequency estimation [4], [5]. Here, by fitting the measured plane phases of the two complex functions to their analytical forms, we are able to estimate the 2-D shifts between two sinusoids.

The 2-D sub-sample estimation is shown to be unbiased for long signals. An iterative version of the estimator is proposed as that the same accuracy for short signals is obtained.

2. MODEL

In this work, we assume that the two signals are shifted versions of a 2-D sinusoid.

$$
\begin{aligned}
r(m,n) &= \cos(2\pi f_1 m)\cdot\cos(2\pi f_2 n)\\
s(m,n) &= \cos[2\pi f_1(m-\Delta_1)]\cdot\cos[2\pi f_2(n-\Delta_2)],
\end{aligned}
\tag{1}
$$

where :
- $m \in [1..M]$ and $n \in [1..N]$
- f_1 and f_2 are the normalized frequencies along the two directions
- Δ_1 and Δ_2 are the shifts to be estimated.

We note $R(k,l)$ the cross-correlation function between the 2-D signals in (1). We consider two complex functions, constructed using the cross-correlation function and its Hilbert transforms [6], noted H_m or H_n in one direction and H_{mn} in both directions ($H_{mn}\{\cdot\} = H_m\{H_n\{\cdot\}\}$).

$$
\begin{aligned}
\tilde{R}_+(k,l) &= R(k,l) - H_{mn}\{R(k,l)\}\\
&\quad + j\cdot(H_m\{R(k,l)\} + H_n\{R(k,l)\})\\
\tilde{R}_-(k,l) &= R(k,l) + H_{mn}\{R(k,l)\}\\
&\quad + j\cdot(H_m\{R(k,l)\} - H_n\{R(k,l)\}),
\end{aligned}
\tag{2}
$$

where $k \in [-M..M]$ and $l \in [-N..N]$.

Analytical calculations show that the phases of the complex functions in (2) have the following forms (modulo 2π):

$$
\begin{aligned}
\phi\left(\tilde{R}_+(k,l)\right) &= 2\pi f_1(k-\Delta_1) + 2\pi f_2(l-\Delta_2)\\
\phi\left(\tilde{R}_-(k,l)\right) &= 2\pi f_1(k-\Delta_1) - 2\pi f_2(l-\Delta_2)
\end{aligned}
\tag{3}
$$

3. METHOD

3.1. Direct estimation

In order to estimate Δ_1 and Δ_2, we propose a new method consisting in a least square plane fitting of the phases of the two complex functions. Mathematically, the plane fitting will be carried out by minimizing the square errors between the theoretical and the measured phases. The two expressions to minimize are:

$$J_+(k,l)=\sum_{k=-M}^{M}\sum_{l=-N}^{N}\left[2\pi f_1(k-\Delta_1)+2\pi f_2(l-\Delta_2)-\phi_+(k,l)\right]^2$$
$$J_-(k,l)=\sum_{k=-M}^{M}\sum_{l=-N}^{N}\left[2\pi f_1(k-\Delta_1)-2\pi f_2(l-\Delta_2)-\phi_-(k,l)\right]^2, \qquad (4)$$

where $\phi_+(k,l)$ and $\phi_-(k,l)$ are the measured phases of the complex functions defined in (2).

By differentiating equations (4) according to Δ_1 (the same results according to Δ_2) and after simplifications, we obtain:

$$\sum_{k=-M}^{M}\sum_{l=-N}^{N}[2\pi f_1(k-\hat{\Delta}_1)+2\pi f_2(l-\hat{\Delta}_2)-\phi_+(k,l)]=0$$
$$\sum_{k=-M}^{M}\sum_{l=-N}^{N}[2\pi f_1(k-\hat{\Delta}_1)-2\pi f_2(l-\hat{\Delta}_2)-\phi_-(k,l)]=0 \qquad (5)$$

After simplifications, the estimated parameters become:

$$\hat{\Delta}_1=\frac{1}{4\pi f_1(2M+1)(2N+1)}\sum_{k=-M}^{M}\sum_{l=-N}^{N}[\phi_+(k,l)+\phi_-(k,l)]$$
$$\hat{\Delta}_2=\frac{1}{4\pi f_2(2M+1)(2N+1)}\sum_{k=-M}^{M}\sum_{l=-N}^{N}[\phi_-(k,l)-\phi_+(k,l)] \qquad (6)$$

We observe now a noisy sequence $R(k,l)$. For simplicity, an additive zero-mean white Gaussian noise $v(k,l)$, with variance σ_v^2, is considered. We note SNR_v the signal-to-noise ratio corresponding to the correlation function.

$$R_v(k,l)=R(k,l)+v(k,l) \qquad (7)$$

The Hilbert transform is linear and it does not change the statistical properties of the noise [6]. The two complex functions defined in (2) are consequently embedded in additive zero-mean complex white Gaussian noise, with variances $4\sigma_v^2$. $A_+(k,l)$ and $A_-(k,l)$ are the magnitudes of the two complex functions defined in (8).

$$\tilde{R}_{+,v}(k,l)=A_+(k,l)\exp\big(j(2\pi f_1(k-\Delta_1)+2\pi f_2(l-\Delta_2))\big)+z_1(k,l)$$
$$\tilde{R}_{-,v}(k,l)=A_-(k,l)\exp\big(j(2\pi f_1(k-\Delta_1)-2\pi f_2(l-\Delta_2))\big)+z_2(k,l) \qquad (8)$$

The variances of real and imaginary parts are $2\sigma_v^2$. The two noises $z_1(k,l)$ and $z_2(k,l)$ are statistically independent.

As shown by Tretter [7] the additive complex noise can be converted into an additive phase noise for high values of SNR_v. This results in the phases $\phi_+(k,l)$ and $\phi_-(k,l)$ being embedded in additive zero-mean Gaussian independent noises.

$$\phi\left(\tilde{R}_{+,v}(k,l)\right)=2\pi f_1(k-\Delta_1)+2\pi f_2(l-\Delta_2)+b_1(k,l)$$
$$\phi\left(\tilde{R}_{-,v}(k,l)\right)=2\pi f_1(k-\Delta_1)-2\pi f_2(l-\Delta_2)+b_2(k,l), \qquad (9)$$

with $\operatorname{var}[b_1(k,l)]=\operatorname{var}[b_2(k,l)]=\dfrac{1}{2SNR_v}$

In these conditions, the estimators in (6) are easily shown to be unbiased for high SNR_v, with variances:

$$\operatorname{var}[\hat{\Delta}_1]=\frac{1}{(4\pi f_1)^2(2M+1)(2N+1)SNR_v}$$
$$\operatorname{var}[\hat{\Delta}_2]=\frac{1}{(4\pi f_2)^2(2M+1)(2N+1)SNR_v} \qquad (10)$$

3.2. Iterative estimation

Theoretical results show that the estimations are unbiased. This assumption is true if we take into account a large number of signal periods. In the cases of small number of periods, even for large signal-to-noise ratios, the estimations defined in (6) are biased. These biases are caused by the fact that we assume now truncated signals, whereas the direct estimation assumed long signals. In usual applications, the number of periods considered cannot be very large. Consequently, we propose a method to eliminate the biases, using a small number of periods: making the estimator iterative. Between two iterations, the shift between the signals is compensated by taking into account the result of the previous estimation. Thus, the shifts we have to estimate are smaller as the iterations advance.

For a given number of periods, the difference between measured and theoretical phases (defined in (3)) becomes smaller if the shift between the signals decreases. Therefore, the estimation will be more accurate if the shifts we want to estimate are smaller, which is true when we advance in the iterations.

Analytically, we assume that the phases defined in (3) have the following forms:

$$\phi^{(i)}(\tilde{R}_+(k,l)) = 2\pi f_1(k-\Delta_1) + 2\pi f_2(l-\Delta_2) + B_+^{(i)}(k,l)$$
$$\phi^{(i)}(\tilde{R}_-(k,l)) = 2\pi f_1(k-\Delta_1) - 2\pi f_2(l-\Delta_2) + B_-^{(i)}(k,l), \quad (11)$$

where $B_+^{(i)}(k,l)$ and $B_-^{(i)}(k,l)$ are the differences of the measured phases compared to the theoretical forms at iteration i.

The shift estimations after i iterations become:

$$\hat{\Delta}_1^{(i)} = \Delta_1 - \frac{\sum_{k=-M}^{M}\sum_{l=-N}^{N} B_+^{(i)}(k,l) + \sum_{k=-M}^{M}\sum_{l=-N}^{N} B_-^{(i)}(k,l)}{4\pi f_1(2M+1)(2N+1)}$$
$$\hat{\Delta}_2^{(i)} = \Delta_2 - \frac{\sum_{k=-M}^{M}\sum_{l=-N}^{N} B_+^{(i)}(k,l) - \sum_{k=-M}^{M}\sum_{l=-N}^{N} B_-^{(i)}(k,l)}{4\pi f_2(2M+1)(2N+1)} \quad (12)$$

The equations in (12) show that the estimations converge towards the true values of the shifts, just as the differences between the two phases and their theoretical forms tend towards zero.

4. COMPUTER SIMULATION RESULTS

4.1. Direct estimation

A computer simulation was performed to show the performance of the estimator. Numerically, we consider:

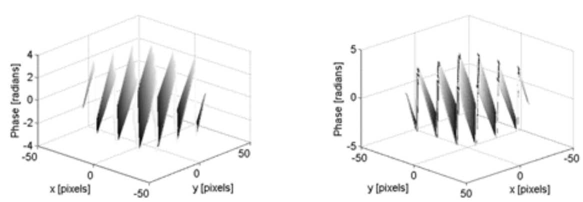
$M = N = 400$

$f_1 = f_2 = 0.05$

$\Delta_1 = 0.15$

$\Delta_2 = 0.25$

Fig. 1 shows the phases of the two complex functions defined in (2), around the values of the shifts.

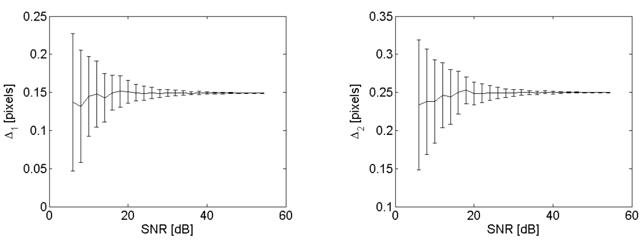


(a) (b)

Fig. 1. Phases of (a) $\tilde{R}_+(k,l)$ and (b) $\tilde{R}_-(k,l)$ around the values of the shifts

We observe that the phases are not linear over the entire domain. It may be possible to unwrap the phases to obtain plane phases [8]. In this work, we extract a plane from each phase and the estimation will use only these two extracted planes. In order to choose the most advatageous planes for our estimations, we extract them around the shifts corresponding to the maximum of the correlation function $R(k,l)$. Note that the precision of this maximum does not influence the further estimations. It serves only to extract the two phase planes.

For this simulated case (corresponding to a signal-to-noise ratio of 50 dB), the estimator gives the exact values of the shifts. The quality of the estimations in the presence of noise, in terms of mean value and of standard deviation is given in Fig. 2.



(a) (b)

Fig. 2. Shift estimations in presence of noise (a) along m and (b) along n

Fig. 3 shows a comparison between the measured and the theoretical variances as functions of SNR.

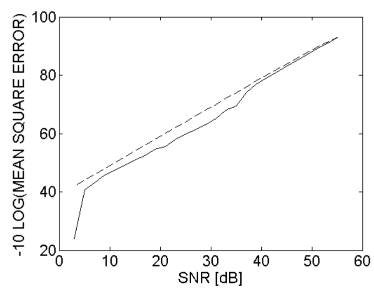
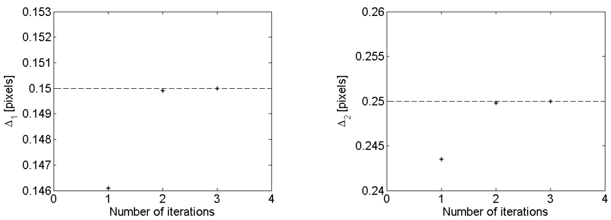


Fig. 3. Theoretical (dashed line) and measured (solid) log-variances as a function of SNR in dB

The measured variance is shown to be closer to the analytical form for high SNR.

4.2. Iterative estimation

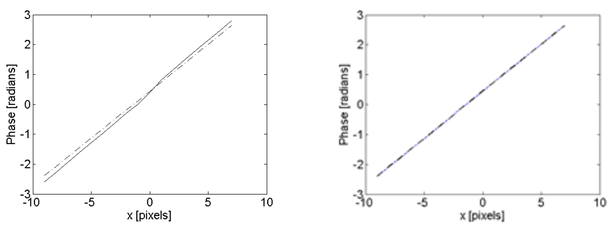
The simulation results of the direct estimation were obtained using 20 periods from each signal. If we consider only two periods and a SNR of 50 dB, the results of the direct estimation are biased, which shows the advantage of the iterative estimation. Fig. 4 shows that in three iterations we converge towards the true values of the shifts, considering truncated two-period signals.



(a) (b)

Fig. 4. Estimations as functions of the number of iterations (*) and true values of the shifts (dashed line) along directions (a) m and (b) n

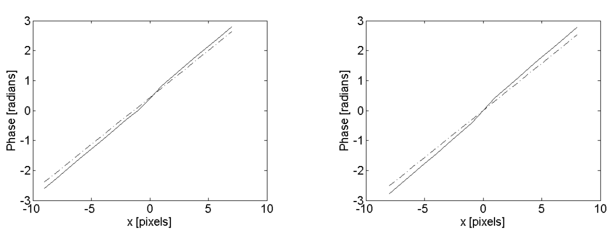
The influence of the number of periods taken into account is shown in Fig. 5. We can see how the measured phase (as a function of x, for a given y) gets closer to the expected form by increasing the number of periods.



(a) (b)

Fig. 5. Measured (solid) and theoretical (dashed line) phases for 2-D signals with (a) 2 and (b) 20 periods

The influence of the value of the shift on the difference between the measured and the theoretical phases is shown in Fig. 6 for a given number of periods. An inflection of the measured phase is visible. The inflection point depends on the value of the shift between the signals. It is closer to zero lag when the shift decreases. In this case, the errors made before and after the inflection point are compensated and the estimation is more accurate.



(a) (b)

Fig. 6. Measured (solid) and theoretical (dashed line) phase for a shift of (a) 0.2 and (b) 0.01 pixels for 2-D signals with two periods

4. CONCLUSION

In this paper, a new method of 2-D sub-sample shift estimation between two sinusoids is proposed. Two complex functions are defined, using the cross-correlation and its Hilbert transforms, in order to have linear phases on both directions. The proposed estimator is based on a plane fitting between the theoretical and measured phases. Simulation results show the performance of the estimator and show its accuracy in presence of additive white Gaussian noise.

An iterative version of the estimator is also proposed for a better estimation with truncated signals. We show how for a small number of periods considered, the estimator gives the true values of the shifts in three iterations.

These findings suggest that the method described in this paper can be used as a 2-D local sub-pixel displacement estimation in ultrasound imaging.

5. REFERENCES

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