LOW REDUNDANCY OVERSAMPLED LAPPED TRANSFORMS AND APPLICATION TO 3D SEISMIC DATA FILTERING

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ABSTRACT

In a previous work, we proposed a relatively simple method to build non separable perfect reconstruction oversampled lapped transforms. The main drawback of this method was that the redundancy factor was constrained to be equal to the overlapping one. This constitutes a strong limitation for applications such as seismic processing involving three-dimensional data sets. The memory requirements may indeed become hard to meet if the redundancy is not reduced. In this paper, we propose an approach to guarantee that a given lapped transform is invertible by a finite length filter bank. We show how to compute a corresponding synthesis filter bank. The proposed analysis/synthesis filter bank system is applied to directional filtering of noisy three-dimensional seismic data.

1. INTRODUCTION

In seismic exploration, arrays of sensors are distributed on the ground or sea surface. Each sensor records a one-dimensional signal. It represents the propagating waves which are reflected or refracted by the interfaces separating different geological strata. Signals are grouped in two- or three-dimensional data sets (images or volumes), each one accounting for one column of a higher dimensional data set [1]. Data acquired from neighboring sensors bear some correlation; as a consequence, seismic data gather local information on the subsurface structures. Since the 1D signals are often oscillatory, it is reasonable to analyze them with localized transforms, capable of catching these oscillations.

Filter banks (FBs) are obvious candidates, the subclass of Lapped Transforms (LTs) being among the most appealing solutions. LTs were primarily aimed at reducing blocking or checkerboard artifacts in audio or image processing. Critically sampled, or non-redundant, LTs are grounded on an extensive literature [2]. They have been applied successfully to audio or image coding [3, 4], image or seismic data denoising.

It has been early recognized that if critical sampling is usually convenient for data compression, it reaches its limitations for data analysis or denoising, for which shift-invariance properties may be of paramount importance. As a consequence, there has been a growing interest in the redundant setting, motivated by attractive properties and specific applications. In particular, oversampled LTs (and FBs in general) offer more robustness with respect to noises [5, 6] and aliasing distortion.

With redundant representations, the degrees of freedom in the filter design increase and inverse transforms are not unique in general. General theoretical analyses and specific reconstruction criteria have been proposed [7, 6], but the design of the reconstruction filter bank remains a difficult task, especially when finite-length inverse filters are looked for; some approaches on direct design have nevertheless been recently proposed [8].

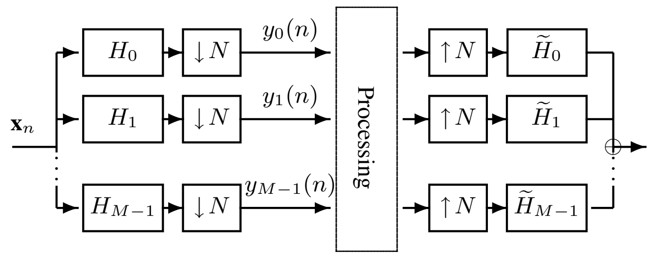


Fig. 1. Oversampled Filter Bank.

In [9], we focused on the design of non-separable complex modulated transform for seismic image denoising and proposed FBs with a relatively high redundancy factor. This one was directly linked to the filter overlapping factor. This approach was tractable for the 2D images which were considered. However, for actual 3D seismic data analysis, the large overlapping factor required to perform a good frequency analysis induces strong limitations in terms of memory load.

In this work, we address the design of oversampled LTs with a reduced redundancy for three-dimensional seismic data filtering. We provide a practical method for the computation of an inverse transform. Since these results are based on a polyphase representation of the FBs [10], fast signal transforms can be applied with filter operations being performed at the downsampled rate.

In Section 2, we first recall basic facts about oversampled FBs and introduce our notations. In Section 3, we present a method for checking the perfect reconstruction property of such structures and, when this condition is satisfied, describe how a synthesis FB can be obtained. In Section 4, so-designed oversampled FBs with reduced redundancy are applied to 3D seismic data filtering. Some conclusions are finally drawn in Section 5.

2. OVERSAMPLED FILTER BANKS

2.1. Notation

We first recall polyphase representation notations. Figure 1 represents the considered 1D M-band filter bank structure which is considered. The signal $(x(n))_{n\in\mathbb{Z}}$ is decomposed using M filters with impulse responses: $(H_i)_{0\leq i<M}$, each one having a finite length kN

with $k \in \mathbb{N}^*$. Then, a decimation of a factor N is performed. From the lapped transform viewpoint, there are therefore $k-1$ overlapping blocks of size N. The M outputs of the analysis FB are denoted by $(y_i(n))_{0\le i<M}$. We are interested in the oversampled case, *i.e.* $N < M = k'N$. The downsampling factor being N, we deduce that the redundancy of the transform is: $M/N = k'$.

Using these notations, the outputs of the analysis FB are expressed, for all $i \in \{0,\dots,M-1\}$ and $n \in \mathbb{Z}$, as

$$\begin{aligned} y_i(n) &= \sum_p H_i(p)x(Nn-p) \\ &= \sum_\ell \sum_{j=0}^{N-1} H_i(N\ell+j)x(N(n-\ell)-j). \end{aligned} \tag{1}$$

Let

$$\boldsymbol{H}(\ell) = (H_i(N\ell+j))_{0\le i<M, 0\le j<N}, \quad \ell \in \{0,\cdots,k-1\}$$

be the k matrices obtained from the impulse responses of the filters. We also define the vector signal:

$$\forall n \in \mathbb{Z}, \qquad \mathbf{x}(n) = (x(Nn-j))_{0\le j<N}$$

from the input signal $x(n)$. A more concise form of Eq. (1) can then be provided:

$$\begin{aligned} \mathbf{y}(n) &= (y_0(n),\dots,y_{M-1}(n))^\top \\ &= \sum_\ell \boldsymbol{H}(\ell)\mathbf{x}(n-\ell) = (\boldsymbol{H} * \mathbf{x})(n), \end{aligned} \tag{2}$$

or, equivalently,

$$\mathbf{y}[z] = \boldsymbol{H}[z]\mathbf{x}[z],$$

where: $\boldsymbol{H}[z] = \sum_{\ell=0}^{k-1} \boldsymbol{H}(\ell) z^{-\ell}$ is the $M\times N$ polyphase transfer matrix of the analysis filter bank and $\mathbf{x}[z]$ and $\mathbf{y}[z]$ are the z-transforms of $(\mathbf{x}(n))_{n\in\mathbb{Z}}$ and $(\mathbf{y}(n))_{n\in\mathbb{Z}}$, respectively.

These expressions hold for any oversampled FB, but we will now provide more details about the actual transform which is investigated in this paper.

2.2. Seismic data and modulated filter banks

Seismic data are highly anisotropic. Performing a directional transform can be very useful to extract their most meaningful structures, which are often embedded in noise. Complex transforms appear well-suited for separating oriented features in 2 or 3 dimensions. Especially in the separable case, real transforms may indeed exhibit undesirable symmetries in the frequency plane (for instance, preventing from separating oriented features with θ or $\frac{\pi}{2}-\theta$ angles in the 2D case). Moreover, the oscillatory nature of seismic data suggests the use of a frequency transform.

Complex valued harmonic analyses such as those performed by DFT (Discrete Fourier Transform) FBs are good candidates for such purposes. In this paper, we choose a decomposition derived from the extended Complex Lapped Transform proposed in [11]. The impulse responses of the filters are then given by: for all $(i,p) \in \{0,\dots,M-1\}\times\{0,\dots,kN-1\}$,

$$H_i(p) = \boldsymbol{E}(i,p)h_a(p), \tag{3}$$

where

$$\boldsymbol{E}(i,p) = \frac{1}{\sqrt{k'N}} e^{-\imath(i-\frac{k'N}{2}+\frac{1}{2})(p-\frac{kN}{2}+\frac{1}{2})\frac{2\pi}{k'N}}, \tag{4}$$

and $(h_a(p))_{0\le p<kN}$ is an analysis window. Its choice often results from a trade-off among its time/frequency localization properties. An example of such a window is:

$$h_a(p) = \sin\Big(\frac{(p+1)\pi}{kN+1}\Big). \tag{5}$$

In our previous work [9], the merits of oversampled DFT FBs have been recognized for seismic data denoising applications. We also established conditions on the analysis window for such FBs to correspond to discrete-time (possibly tight) frame decompositions and we derived the associated pseudo-inverse operators for optimal reconstruction. This study was however restricted to the case when $k' = k$. This relation between the redundancy factor and the overlapping one appears restrictive for two main reasons. At first, as soon as a non-instantaneous transform is employed, the redundancy factor has to be greater than or equal to 2, which may become prohibitive for processing large data sets. Secondly, we lose the ability of adjusting independently these two parameters so as to provide a flexible analysis. Furthermore, the matrix formulation adopted in [9] seems inappropriate to address the case when $k' \neq k$. In order to build modulated FBs with low redundancy while keeping good analyzing and perfect reconstruction properties, we will now resort to algebraic results from the theory of MIMO systems [12].

3. EXISTENCE AND DETERMINATION OF A FIR SYNTHESIS FB

3.1. Invertibility of the analysis FB

The polyphase representation of filter banks offers the advantage of relating the perfect reconstruction property to the invertibility of the polyphase transfer matrix [10]. The matrix under investigation belongs to the ring $\mathbb{C}[z,z^{-1}]^{M\times N}$ of Laurent polynomial matrices of dimensions $M \times N$. We emphasize that we are not looking for any inverse MIMO filter but for an inverse *polynomial* matrix in $\mathbb{C}[z,z^{-1}]^{N\times M}$. In other words, we aim at obtaining a Finite Impulse Response (FIR) synthesis FB.

The following theorem [13] is the cornerstone of the proof for the existence of such an inverse system:

Theorem 1 *Let $\boldsymbol{H}[z] \in \mathbb{C}[z,z^{-1}]^{M\times N}$ be a polynomial matrix with $M > N$. The following conditions are equivalent:*

1. *$\boldsymbol{H}[z]$ is "coprime", which means that its maximum minor determinants are mutually relatively prime.*
2. *$\boldsymbol{H}[z]$ is left invertible in the sense that there exists $\boldsymbol{M}[z] \in \mathbb{C}[z,z^{-1}]^{N\times M}$ such that $\boldsymbol{M}[z]\boldsymbol{H}[z] = \boldsymbol{I}_N$.*

The first condition of this theorem is directly applicable to find out whether the polyphase transfer matrix is left invertible or not. A closer inspection reveals however that it is difficult to expect more focused results without using numerical techniques. More precisely, the following procedure can be followed to check whether the first condition is satisfied:

① Extract a maximal sub-matrix $\boldsymbol{H}_e[z]$ of $\boldsymbol{H}[z]$.

② Compute $\det(\boldsymbol{H}_e[z])$, and determine the set $\mathcal{S}_e$ of its roots.

③ Consider another maximal sub-matrix. Remove from $\mathcal{S}_e$ the elements which are not roots of the determinant of this sub-matrix.

④ Repeat step ③ until all maximal sub-matrices have been extracted or $\mathcal{S}_e = \emptyset$.

⑤ If $\mathcal{S}_e = \emptyset$ then the polyphase transfer matrix is left-invertible; otherwise, it is not.

The corresponding algorithm is easily implemented, yielding to check the roots of at most $\binom{N}{M}$ polynomials. Notice that this method allows to guarantee the existence of a left-inverse but not to determine its expression. We will see in Section 3.2 how to perform the computation of an inverse polyphase transfer matrix.

Example 1 Let us illustrate this algorithm with a very simple example: we take $N = 2$, $k = 2$, $k' = 3/2$ and the LT defined by (3)-(5). The polyphase matrix $\boldsymbol{H}[z]$ is thus in $\mathbb{C}[z, z^{-1}]^{3\times 2}$. To check that this polynomial matrix is left invertible by a matrix in $\mathbb{C}[z, z^{-1}]^{2\times 3}$, we have to consider the maximal sub-matrices of $\boldsymbol{H}[z]$:

$$\boldsymbol{H}_\ell[z] = \left((\boldsymbol{H}[z])_{j,k}\right)_{(j,k)\in A_\ell \times\{1,2\}}, \quad \ell \in \{1, 2, 3\}$$

with $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$ and $A_3 = \{1, 3\}$. Since $k = 2$, the elements of the matrix $\boldsymbol{H}[z]$ are of degree 1 and the degree of $\det(\boldsymbol{H}_\ell[z])$ is 2 at most. Table 1 gives the two roots for each of the polynomials $\det(\boldsymbol{H}_\ell[z])$. We see that there is no common root to the three determinants. Consequently, the matrix $\boldsymbol{H}[z]$ is left invertible by a matrix in $\mathbb{C}[z, z^{-1}]^{2\times 3}$.

	First root	Second root
$\det(\boldsymbol{H}_1[z])$	$0.1045 - 0.9945\imath$	$-0.9135 - 0.4067\imath$
$\det(\boldsymbol{H}_2[z])$	$0.1045 - 0.9945\imath$	$0.9135 + 0.4067\imath$
$\det(\boldsymbol{H}_3[z])$	$0.8090 + 0.5878\imath$	$0.8090 - 0.5878\imath$

Table 1. The roots of the three minor determinants of $\boldsymbol{H}[z]$.

3.2. Computation of the inverse polyphase transfer matrix

In this section, we suppose that by applying the previous method, the polyphase transfer matrix $\boldsymbol{H}[z]$ was found to be coprime, hence left invertible by a matrix $\boldsymbol{M}[z] \in \mathbb{C}[z, z^{-1}]^{N\times M}$. Therefore, we can claim that there exists $q \in \mathbb{N}^*$ such that $\boldsymbol{M}[z] = \sum_{\ell=1-q}^{0} \boldsymbol{M}(\ell)z^{-\ell}$ and

$$\boldsymbol{M}[z]\boldsymbol{H}[z] = \boldsymbol{I}_N. \tag{6}$$

This matrix remains however to be computed.
We obviously have

$$\boldsymbol{M}[z]\boldsymbol{H}[z] = \sum_{\ell=1-q}^{0} \boldsymbol{M}(\ell)z^{-\ell} \sum_{\ell=0}^{k-1} \boldsymbol{H}(\ell)z^{-\ell} = \sum_{\ell=1-q}^{k-1} \boldsymbol{U}(\ell)z^{-\ell},$$

where

$$\boldsymbol{U}(\ell) = \sum_{s=\max(\ell-k+1,1-q)}^{\min(0,\ell)} \boldsymbol{M}(s)\boldsymbol{H}(\ell - s).$$

Equation (6) is then equivalent to $\boldsymbol{U}(\ell) = \delta_\ell \boldsymbol{I}_N$, which leads to the following linear equation:

$$\mathcal{H}\mathcal{M} = \mathcal{U}$$

where

$$\mathcal{M}^\top = [\boldsymbol{M}(1 - q), \cdots, \boldsymbol{M}(0)],$$
$$\mathcal{U}^\top = \left[\boldsymbol{0}_{N,(q-1)N}\ \boldsymbol{I}_N\ \boldsymbol{0}_{N,(k-1)N}\right],$$

and

$$\mathcal{H}^\top = \begin{pmatrix} \boldsymbol{H}(0) & \cdots & \boldsymbol{H}(k-1) & & 0 \\ & \ddots & & \ddots & \\ 0 & & \boldsymbol{H}(0) & \cdots & \boldsymbol{H}(k-1) \end{pmatrix}.$$

The value of q being unknown *a priori*, we have to try to solve the above system for increasing values of q in order to find the minimum order of an inverse polyphase transfer matrix candidate.

Example 2 If we apply this method to the FB considered at the end of the previous section, we find $q = 2$.

4. APPLICATION TO 3D SEISMIC DATA

4.1. Three-dimensional filtering

Since we are working with 3D volumes, a three-dimensional separable FB is built by a tensor product of three monodimensional FBs as defined in Section 2.2. Depending on the anisotropy present in the data, it is possible to adapt the resulting 3D FB by selecting 1D FBs with appropriate characteristics. Since the proposed method relies on the polyphase representation, filtering occurs after downsampling, so reducing drastically the computational cost.

Once the 3D filter is designed, it can be applied to the removal of directional noises and the enhancement of important geological structures. The subband coefficients are then processed by local non-linear operators reminiscent of thresholding estimators. More precisely, we first locate the locally dominant direction with the coefficients of maximum magnitude. Coefficients outside a band in the time-frequency plane corresponding the local direction are cancelled, while remaining noisy coefficients undergo a hard thresholding stage. Finally, the signal is reconstructed using a corresponding synthesis FB.

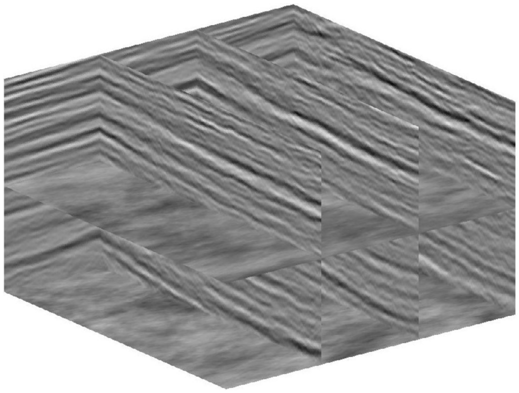
In the toy example provided in Section 3.1, values taken for N and k were small. These parameters require to be adapted to the nature of the data, e.g. the sampling step for each dimension. In the following simulations, we choose $N = 16$ with an overlapping factor $k = 3$. We notice that a high redundancy is usually desirable for improving denoising performance. In this application however, it may severely increase the memory burden due to high dimensionality of the data set.

4.2. Results

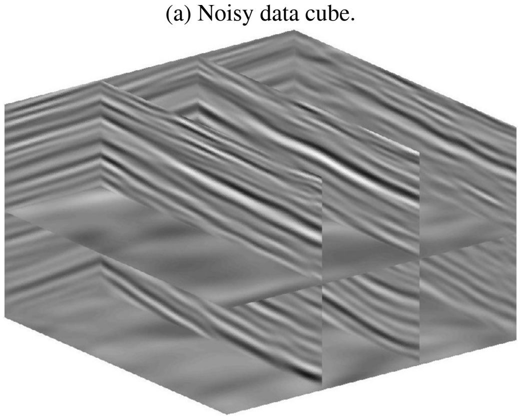
To evaluate the quality of the filtering, the results of this approach (with $k = 3$ and $k' = 3/2$) were compared to those obtained by extending the method in [9] in three dimensions with $k = k' = 3$. The visual results of the two methods are very close despite the different redundancy factors: in this three-dimensional case the redundancy becomes $3^3 = 27$ using the extension of the method in [9] and $27/8$ for the proposed approach.

Then, the proposed method was applied to actual seismic data. On Figure 2-a, a typical seismic data cube is represented via some horizontal and vertical slices. Directional noise intersects the quasi-horizontal layers. Its filtered version is shown in Figure 2-b.

Filtering performance is better visualized on two-dimensional details of the cubic slices, as shown in Figure 3. We see that the directional interferences are well removed while preserving the dominant, almost horizontal layered structures.



(a) Noisy data cube.



(b) Filtered data cube.

Fig. 2. (a) A sample of a 3D seismic data cube, (b) Filtered data cube.

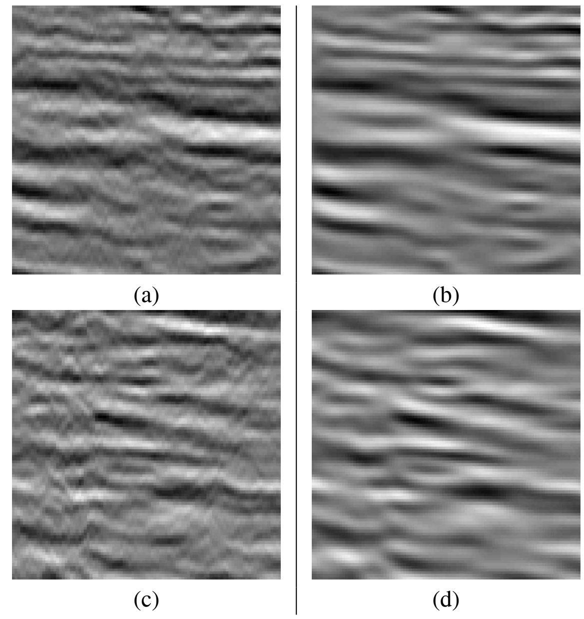


Fig. 3. (a)(c) Slices of original 3D data cube, (b)(d) Corresponding slices of the reconstructed cube.

5. CONCLUSIONS

In this work, an approach was proposed to ensure that a given oversampled LT is invertible by a finite length FB, as well as a method to compute such a synthesis FB. We successfully applied these analysis tools on low redundancy DFT based LTs and then used these FBs for seismic 3D data directional filtering. A first interesting extension of this work would be to try to apply these methods to other classes of LTs. We would also like to find criteria to constrain the synthesis FB to have better properties (for instance improved decay as in [3]). Given that the proposed filtering method removes many coefficients in the frequency space, perfect reconstruction could also be relaxed to near perfect reconstruction, thus allowing more freedom in the filter design (aiming at higher stopband attenuation and reduced aliasing).

6. REFERENCES

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