PUPIL CONFIGURATION FOR EXTENDED SOURCE IMAGING WITH OPTICAL INTERFEROMETRY: A COMPUTATIONAL GEOMETRY APPROACH

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ABSTRACT

The input pupil of interferometry-based observation instruments is necessarily segmented. Since the Optical Transfer Function (OTF) of any optical instrument observing in incoherent light is the auto-correlation of its input pupil, it follows that any combination of size and position of each pupil segment will have an impact on the OTF behavior and therefore on the quality of the output image. The goal of this study is to propose computational geometry methods allowing to find pupil geometries leading to an isotropic OTF support with a controlled redundancy of viewed spatial frequencies in the Fourier domain.

1. INTRODUCTION

The diameter of the primary mirror of a telescope is proportional to its resolution power. In order not to build too large mirrors, interferometric telescopes [1] have been adopted as they synthesize (very) large instruments by interferometrically combining several smaller instruments (also called pupils). Such a method is more specifically called Optical Aperture Synthesis (OAS) and is used in astronomy from Earth.

Let us now imagine observing the Earth from a high orbit (e.g., at a distance of $\sim$ 36000 km) with a Ground Sample Distance (GSD) of 1 m. A simple calculus shows us that we would need a telescope having a diameter of approximately 20 m for an optical wavelength $\lambda \sim 500$ nm. Needless to say, such an instrument would not be adapted to the observation from space and the use of OAS is again to be considered in this case.

Given a set $\mathcal{P}$ of circular pupils of the OAS instrument in a plane, the Optical Transfer Function (OTF) of this multi-pupil set is the auto-correlation [2] of $\mathcal{P}$. The support of the OTF denotes all the observable spatial frequency components. Indeed, the output of each of the individual pupils is interferometrically combined with each other and each interferometer pair measures a "Fourier domain" bounded by the intercorrelation support (ICS) of two pupils.

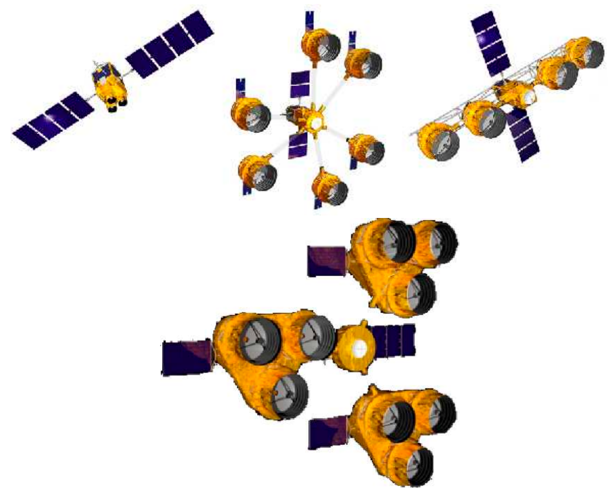

Fig. 1. Examples of OAS spaceborne instrument concepts [3]

Furthermore, for a wavelength λ, a GSD p and a platform altitude h, the auto-correlation support (ACS) of the pupils should cover a square of side $2c = \lambda h/p$ (the sampling frequency) centered at the origin. A good approximation consists in covering the disk of diameter $2c$ inscribed in this square while minimizing the Fourier components that lie outside this square in order to reduce aliasing effects.

The underlying problem can be formally stated in geometric terms as follows. Given an objective O supposed to be a disk, design a system of circular pupils $\mathcal{P} = \{P_1, \ldots, P_n\}$ such that its auto-correlation support $\mathcal{C}$ covers entirely the objective while minimizing some cost function. Here $\mathcal{C} = \{t \in \mathbb{R}^2 \mid (\mathcal{P} + t) \cap \mathcal{P} \neq \emptyset\}$ and $\mathcal{P} + t = \{p + t \mid p \in \mathcal{P}\}$. The cost function may include the number of pupils, their positions, their radii, etc.

The outline of this paper follows. In section 2, we introduce power diagrams which play a central role in our study, and use

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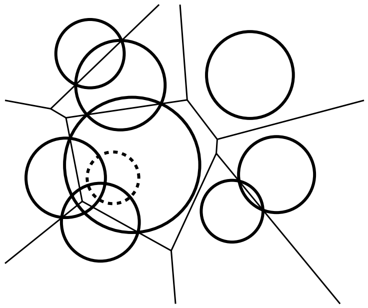

Fig. 2. A power diagram of 9 disks in the Euclidean plane. The dotted circle has no power cell.

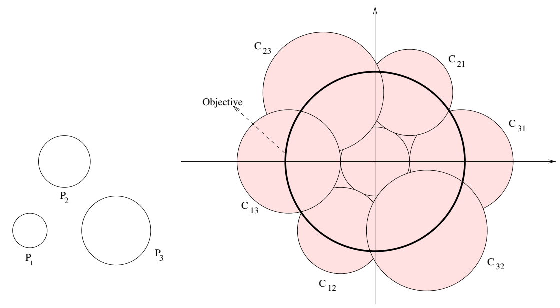

Fig. 3. A system of three pupils (left) and its ACS (right), the objective is represented by a thick circle.

them to decide whether the objective is covered. In section 3, we consider the centers of the pupils to be given and provide an efficient algorithm to minimize the sum of the radii of the pupils under the constraint that the ACS covers the objective. We devote section 4 to considering the problem where the radii of the pupils are known but their positions are unknown. Section 5 concludes the paper.

2. POWER DIAGRAMS AND THE DECISION PROBLEM

Definition. Let $\mathcal{S} = \{D_1, \ldots, D_m\}$ be a set of m disks in the plane. We denote by c_i the center of D_i and by ρ_i its radius. The *power distance* of a point x to the circle ∂D_i is defined as

$$\pi_i(x) = ||x - c_i||^2 - \rho_i^2.$$

For a point x, $\pi_i(x)$ is $< 0, 0, > 0$ depending whether x lies inside, on the boundary of, or outside D_i.

The *power cell* of D_i consists of the points whose power distance to ∂D_i is smaller than their powers to the other circles of $\mathcal{S}$:

$$V_i = \{x \in \mathbb{R}^2 \mid \pi_i(x) \leq \pi_j(x), j = 1, \ldots, m\}.$$

The power cell V_i is the intersection of $m-1$ half-planes. Hence V_i is a convex polygon, possibly empty or unbounded. The collection of all non-empty V_i forms the *power diagram* of $\mathcal{S}$, denoted by $PD(\mathcal{S})$ (see Fig. 2). The edges and the vertices of the power cells are called the edges and the vertices of $PD(\mathcal{S})$. Observe that an edge is incident to two cells and a vertex is, in general, incident to three cells.

The power diagram $PD(\mathcal{S})$ can be computed in time $\mathcal{O}(m \log m)$, which is worst-case optimal, and robust and efficient implementations exist [4].

Decision problem. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a system of n disks called the *pupils* and O be a disk centered at the origin called the *objective*. For $i = 1, \ldots, n$, we denote by c_i and ρ_i the center and the radius of pupil P_i. The ACS of the system is $\mathcal{C} = \{t \in \mathbb{R}^2 \mid (\mathcal{P}+t) \cap \mathcal{P} \neq \emptyset\}$, where $\mathcal{P}+t = \{p+t \mid p \in \mathcal{P}\}$. The decision problem consists in determining whether O is covered by $\mathcal{C}$.

The inter-correlation support of two pupils P_i and P_j is the Minkowski difference $C_{ij} = \{p_i - p_j \mid p_i \in P_i \text{ and } p_j \in P_j\}$ of the P_i and P_j. It is not difficult to see that C_{ij} is a disk with center $c_{ij} = c_i - c_j$ and radius $\rho_{ij} = \rho_i + \rho_j$. Moreover, $\mathcal{C} = \bigcup_{ij} C_{ij}$ (see Fig. 3). We write V_O for the power cell of O and V_{ij} for the power cell of C_{ij} in the power diagram of $\mathcal{C} \cup O$. The two following lemmas show a necessary and sufficient condition for covering O by $\bigcup_{ij} C_{ij}$ (see Fig. 4).

Lemma 1. *$\partial O \subset \operatorname{int} \mathcal{C}$ iff ∂O does not intersect V_O.*

Indeed, if $\partial O \subset \operatorname{int} \mathcal{C}$, then for any point $p \in \partial O$, p lies strictly inside some disk of $\mathcal{C}$, say, C_{ij}, which implies $\pi_O(p) = 0 > \pi_{ij}(p)$. Thus p does not belong to V_O by definition. The sufficient condition is proved similarly.

We need now to check that O does not cover a hole H of $\mathcal{C}$. H being compact, the power distance to $\mathcal{C}$ which is continuous, positive inside H and 0 on ∂H, has to reach a maximum at a point $h \in H$. The point where such a maximum is reached is a vertex of $PD(\mathcal{C})$.

Lemma 2. *Suppose $\partial O \subset \operatorname{int} \mathcal{C}$, O is covered by $\operatorname{int} \mathcal{C}$ iff there is no vertex of $PD(\mathcal{C})$ in $O \setminus \mathcal{C}$.*

Lemmas 1 and 2 give us a simple algorithm that solves the decision problem in $\mathcal{O}(n^2 \log n)$ time.

3. FIXED-CENTER PROBLEM

In this section, the centers of the pupils are fixed and we present two algorithms for optimizing the radii. Both algorithms are based on the following simple observation. Let p denote a point, C a circle of center c and radius r, and $\pi(p)$ the power of p to C. Then the circle of center c and squared radius $r^2 + \pi(p)$ passes through p.

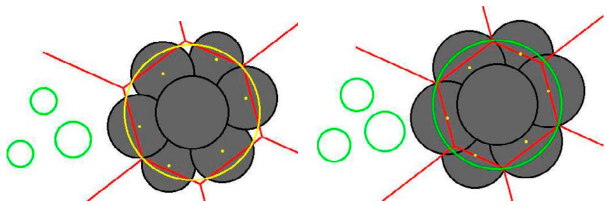

Fig. 4. LEFT: A set of three pupils. Some vertices of V_O in $PD(\mathcal{C} \cup O)$ are outside O and the objective is not covered. RIGHT: The same set of pupils but with squared radii increased by some $\alpha > 0$. All vertices of V_O move towards the origin and are inside O. The objective is covered then.

3.1. A simple optimization problem

In this subsection, we increase/decrease the squared radii (also called weights) of the pupils by a same amount. This leads to an optimization problem in one variable. More precisely, consider adding to each of the weights of the pupils a real number α. This does not change $PD(\mathcal{C})$ and the only vertices of $PD(\mathcal{C} \cup O)$ that need to be modified are the vertices of V_O (see Fig. 4). Hence there exists a minimum value of α for which the hypotheses of Lemmas 1 and 2 hold. Equivalently, this is the smallest α that makes the objective covered by the union of the disks. The following procedure computes such a value of α. By $\pi_{ij}(x)$, we denote the power distance of x to ∂C_{ij}. V_{ij} denotes the power cell of C_{ij} in $PD(\mathcal{C})$.

```
function ALPHA(O, C)
1  α ← −∞
2  compute PD(C)
3  for each vertex p in PD(C) ∩ O
4      // V_ij is a power cell incident to p
5          α ← max(α, π_ij(p))
6  compute PD(C ∪ O)
7  for each vertex q of V_O in PD(C ∪ O)
8      α ← max(α, π_0(q))
9  return α
```

In Lines 2-5, we compute the minimum α such that all vertices of $PD(\mathcal{C}) \cap O$ belong to $\mathcal{C}$. Lines 6-8 compute the minimum value of α that makes all vertices of V_O in $PD(\mathcal{C} \cup O)$ belong to $\mathcal{C}$. The time complexity of the algorithm is dominated by the complexity of Lines 2 and 6, which is $\mathcal{O}(n^2 \log n)$.

Maximizing the objective: Using a similar approach, we obtain a similar result for maximizing the radius of O while keeping the pupils fixed.

3.2. Minimizing the sum of the radii of the pupils

We consider now the more difficult problem of optimizing the sum of the radii of the pupils and propose a heuristic solution that turns out to perform well in practice. Instead of increasing/decreasing the weights by a same amount as in the previous subsection, we consider the radii of the P_i as n variables. Function ALPHA*$(O, \mathcal{C})$ below proceeds in two main steps (see Fig. 5). First, we compute quantities, denoted α_{ij}, by which the squared radii of (some of) the C_{ij} must be enlarged so as to satisfy Lemmas 1 and 2 (lines 1-11). This step is similar to Function ALPHA$(O, \mathcal{C})$. Since the diagrams may change, we iterate this step until the α_{ij} do not increase. Thanks to the fact that the radii of the C_{ij} necessarily increase, it can be shown that the procedure always terminates. Moreover, upon termination, Lemmas 1 and 2 will be satisfied. We then minimize the sum of the radii of the P_i under n^2 constraints (line 12):

$$\begin{array}{lll} \min & \displaystyle\sum_{i=1}^{n} \rho_i & \\ \text{s.t.} & \rho_i + \rho_j \geq \sqrt{\rho_{ij}^2 + \alpha_{ij}}, & i, j = 1, \ldots, n. \end{array}$$

This is a linear program and the feasible set is non-empty. Thus, there exists an optimal solution.

```
function ALPHA*(O, C)
1  α_ij ← −∞,     i, j = {1, ..., n}
2  repeat
3     compute PD(C)
4     for each vertex p of PD(C) ∩ O
5        for each C_ij whose power cell is incident to p
6           α_ij ← max(α_ij, π_ij(p))
7     compute PD(C ∪ O)
8     for each vertex q of V_O in PD(C ∪ O)
9        for each C_ij whose power cell is incident to q
10          α_ij ← max(α_ij, π_ij(q))
11 until no α_ij increases
12 compute {ρ_i}_{i=1,...,n} by solving the above linear program
13 return {ρ_i}_{i=1,...,n}
```

Additional constraints: In addition to covering the objective, we can also bound the radii of the pupils and forbid any overlap among the pupils. This can be done by adding the following constraints to the linear program above

$$\begin{array}{ll} \rho_i + \rho_j \leq ||c_i - c_j||, & 1 \leq i < j \leq n, \\ min\_radius \leq \rho_i \leq max\_radius, & i = 1, \ldots, n. \end{array}$$

4. PROBLEM WITH FIXED AND IDENTICAL RADII

In [5], Golay described an algorithm to search for point arrays with non-redundant ACS (no distinct pairs of pupils have the same inter-correlation support). In our case, the points are the centers of the pupils and redundant arrays are acceptable. We adopt Golay's algorithm with some modifications. First, we discretize the plane into a regular triangular grid $\mathcal{G}$. If the sides of the grid have length ρ, then the disks centered at its vertices with the same radius $\rho/\sqrt{3}$ are sufficient to cover completely the grid. Let S be the minimum set of vertices

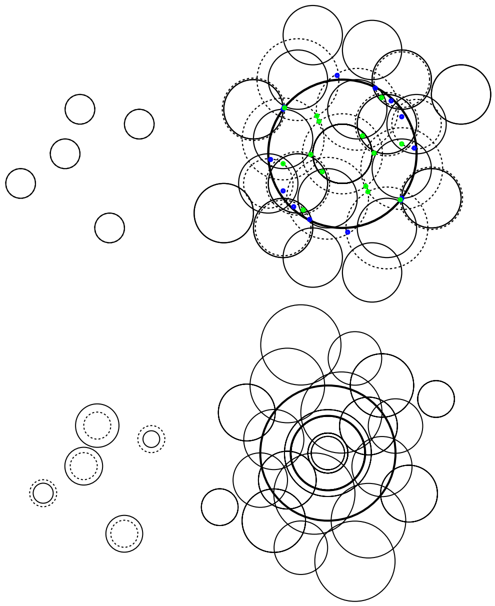

Fig. 5. TOP: A set of 5 unit pupils (left) and its ACS (right) does not cover the objective of radius 5. The dotted circles indicate the disks C_{ij} after increasing the squared radii by α_{ij}. The dark points are the vertices of V_O in $PD(\mathcal{C} \cup O)$ while the gray ones represent the vertices of $PD(\mathcal{C})$ in O. BOTTOM: The same set of pupils P_i with radii increased by α_i (left) and its ACS (right) covers completely the objective. The original pupil set is drawn dotted.

of $\mathcal{G}$ such that the union of the corresponding disks contains O. Our algorithm works as follows: We add the pupils on the grid one by one. We find the vertex of S furthest from the origin o (the center of O) and try to place the next pupil to cover this vertex. We consider all possible cases such that the ICS of this pupil with one of the existing pupils cover the vertex. Next, we place the pupil at the position where its ICS with other pupils covers as many elements in S as possible and then remove these elements from S. We repeat this construction until S is empty. A comparison of our algorithm with Golay's algorithm is shown in Fig. 6. Below is the pseudocode of our algorithm.

function GREEDY(O)
construct a triangular grid $\mathcal{G}$ and compute the set S
$\mathcal{P} \leftarrow \{o\}$ `// pupil center set`
while $S \neq \emptyset$ **do**
  $furthest \leftarrow \arg\max_{p \in S} ||p - o||$
  for each $p \in \mathcal{G}$ such that
  $\exists q \in \mathcal{P} \mid p - q = furthest$ or $q - p = furthest$
    $count_p = \#\{q \in \mathcal{P}, p - q \in S$ or $q - p \in S\}$
  $candidate \leftarrow \arg\max_{p \in \mathcal{G}} count_p$
  for each $q \in \mathcal{P}$
    if $(candidate - q) \in S$ Pop $(candidate - q)$ from S
    if $(q - candidate) \in S$ Pop $(q - candidate)$ from S
  $\mathcal{P} \leftarrow \mathcal{P} \cup \{candidate\}$
return $\mathcal{P}$

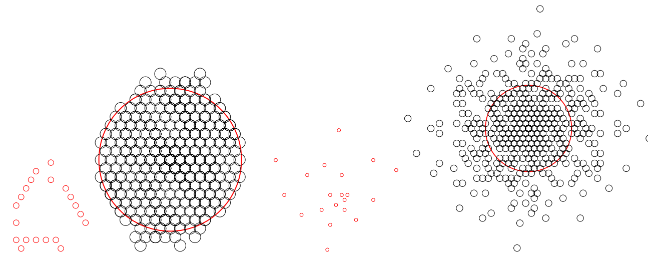

Fig. 6. A set of 20 pupils of unit radius with its corresponding ACS and an objective disk of radius 25. The objective is completely covered in our algorithm (left) while there are some small holes inside the objective in Golay's algorithm (right).

5. CONCLUSION

We have considered the problem of determining the positions and the sizes of pupils to cover an objective in the Fourier domain (the auto-correlation domain). In this paper, the underlying geometric problem is formulated and some initial but efficient algorithms have been presented. This study opens many perspectives of which the problem of arranging a set of pupils with different radii is still unsolved. Moreover, for deconvolution purposes, it remains to optimize jointly the support (the Fourier coverage) and the minimum value of the auto-correlation module, i.e. the modulation transfer function of the instrument.

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6. REFERENCES

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