LINEAR ESTIMATION OF SEQUENCES OF MULTI-DIMENSIONAL AFFINE TRANSFORMATIONS

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ABSTRACT

We consider the general framework of planar object registration and tracking. Given a sequence of observations on an object, subject to an unknown sequence of affine transformations of it, our goal is to estimate the deformation that transforms some pre-chosen representation of this object (template) into the current sequence of observations. We propose a method that employs a set of non-linear operators to replace the original high dimensional and non-linear problem by an *equivalent linear problem*, expressed in terms of the unknown affine transformation parameters. We investigate two modelling and estimation solutions: The first, estimates the affine transformation relating any two consecutive observations, followed by a least squares fit of a global model to the estimated sequence of instantaneous deformations. The second, is a global solution that fits a time-dependent affine model to the entire set of observed data.

1. INTRODUCTION

This paper is concerned with the general problem of object registration and tracking. Given a sequence of observations on an object, subject to an unknown sequence of affine transformations of it, our goal is to track the deformation that transforms some pre-chosen representation of this object into the observed sequence.

To enable a rigorous treatment of the problem we begin by defining the "similarity criterion". Let G be a group and S be a set (a function space in our case), such that G acts as a transformation group on S. The action of G on S is defined by $G \times S \rightarrow S$ such that for every $\phi \in G$ and every $s \in S$, $(\phi, s) \rightarrow s \circ \phi$ (composition of functions on the right), where $s \circ \phi \in S$. From this point of view, given two functions h and g on the same orbit, the basic task is to find the element ϕ in G that makes h and g identical in the sense that $h = g \circ \phi$.

In this paper we concentrate on parametric modelling and estimation of the object deformation as a function of time, based on a sequence of observations on the deforming object. Thus in the current framework the group G is the affine group, and we seek to track the evolution in time of the sequence of affine transformations the observed object undergoes.

2. ESTIMATION OF MULTIDIMENSIONAL AFFINE TRANSFORMATIONS: PROBLEM DEFINITION AND THE BASIC SOLUTION

The basic problem addressed in this section is the following: Given two bounded, Lebesgue measurable functions h, g with compact supports (and with no affine symmetry, as rigorously defined below) such that $h : R^n \rightarrow R$, $g : R^n \rightarrow R$ where

$$h(\mathbf{x}) = g(\mathbf{Ax}) , \quad \mathbf{A} \in GL_n(R), \ \mathbf{x} \in R^n \tag{1}$$

find the matrix $\mathbf{A}$.

Let $M(R^n, R)$ denote the space of compact support, bounded, and Lebesgue measurable functions from R^n to R. Let $N \subset M(R^n, R)$ denote the set of measurable functions with an affine symmetry (or affine invariance), *i.e.*, $N = \{f \in M(R^n, R) | \exists \mathbf{A} \in \mathrm{GL}_n(R), \mathbf{A} \neq \mathbf{I} \text{ such that} f(\mathbf{x}) = f(\mathbf{Ax}) \text{ for every } \mathbf{x} \in R^n\}$. Let $M_{Aff}(R^n, R) \triangleq M(R^n, R) \setminus N$ denote the set of compact support and bounded Lebesgue measurable functions with no affine symmetry. Clearly $M_{Aff}(R^n, R)$ is closed with respect to the affine group operation, hence, if $g \in M_{Aff}(R^n, R)$ then its entire orbit is also is $M_{Aff}(R^n, R)$.

We next provide a constructive proof showing that given an observation $h(\mathbf{x}) \in M_{Aff}(R^n, R)$ and an observation on $g(\mathbf{x}) \in M_{Aff}(R^n, R)$ where $h(\mathbf{x}) = g(\mathbf{Ax})$, $\mathbf{A}$ can be *uniquely* determined.

Let $\mathbf{x}, \mathbf{y} \in R^n$, *i.e.*, $\mathbf{x} = [x_1, \ldots, x_n]^T$, $\mathbf{y} = [y_1, \ldots, y_n]^T$. Thus,

$$\mathbf{y} = \mathbf{Ax} \ \ , \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{A}^{-1} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

Since $\mathbf{A} \in GL_n(R)$, also $\mathbf{A}^{-1} \in GL_n(R)$. It is therefore possible to solve for $\mathbf{A}^{-1}$ and the solution for $\mathbf{A}$ is guaranteed to be in $GL_n(R)$. Moreover, as shown below, in the proposed procedure the transformation Jacobian is evaluated first, and by a different procedure than the one employed to estimate the elements of $\mathbf{A}^{-1}$. Hence, a non-zero Jacobian guarantees the existence of an inverse to the transformation matrix.

Let $f \in M_{Aff}(R^n, R)$ and let μ_n denote the Lebesgue measure on R^n. Define the notation

$$\int_{R^n} f \triangleq \int_{R^n} f d\mu_n$$

Note that in the following derivation it is assumed that the functions are bounded and have compact support, as they are measurable but not necessarily continuous. It is further assumed that $\mathbf{A} \in GL_n(R)$ has a positive determinant.

The first step in the solution is to find the Jacobian of the linear transformation $\mathbf{A}$. Applying some Lebesgue measurable, left-hand composition $w_k : R \to R$ to the known relation $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$ and integrating over both sides of the equality, we obtain

$$\int\limits_{R^n} w_k \circ h(\mathbf{x}) = \int\limits_{R^n} w_k \circ g(\mathbf{A}\mathbf{x}) = |\mathbf{A}^{-1}| \int\limits_{R^n} w_k \circ g(\mathbf{y}) \quad (3)$$

Since $h(\mathbf{x})$ and $g(\mathbf{x})$ are given, while w_k is our choice, all the terms on the LHS and RHS of (3) can be evaluated and hence (3) can be solved for $|\mathbf{A}^{-1}|$.

Next it is shown that, provided that g is "rich" enough in a sense we rigorously define below, $\mathbf{A}$ can be *uniquely* estimated by establishing a system of linear equations in the n unknown elements in each of its rows. More specifically, let $(\mathbf{A})_k$ denote the kth row of $\mathbf{A}$. Applying a family of Lebesgue measurable, left-hand compositions $\{w_\ell\} : R \to R$ to the known relation $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$ and integrating over both sides of the equality, we obtain

$$\begin{aligned} \int\limits_{R^n} x_k w_p \circ h(\mathbf{x}) &= \int\limits_{R^n} x_k w_p \circ g(\mathbf{A}\mathbf{x}) \\ &= |\mathbf{A}^{-1}| \int\limits_{R^n} ((\mathbf{A}^{-1})_k \mathbf{y}) w_p \circ g(\mathbf{y}) \\ &= |\mathbf{A}^{-1}| \sum_{i=1}^{n} q_{ki} \int\limits_{R^n} y_i w_p \circ g(\mathbf{y}) \\ & \qquad\qquad p = 1, \ldots, P \quad (4) \end{aligned}$$

Let

$$\mathbf{G} = \begin{bmatrix} \int\limits_{R^n} y_1 w_1 \circ g(\mathbf{y}) & \cdots & \int\limits_{R^n} y_n w_1 \circ g(\mathbf{y}) \\ \ddots & \vdots & \\ \int\limits_{R^n} y_1 w_P \circ g(\mathbf{y}) & \cdots & \int\limits_{R^n} y_n w_P \circ g(\mathbf{y}) \end{bmatrix}$$

Rewriting (4) in a matrix form

$$\mathbf{G} \begin{bmatrix} q_{k1} \\ \vdots \\ q_{kn} \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| \int\limits_{R^n} x_k (w_1 \circ h(\mathbf{x})) \\ \vdots \\ |\mathbf{A}| \int\limits_{R^n} x_k (w_P \circ h(\mathbf{x})) \end{bmatrix} \quad (5)$$

Provided that the sequence of composition functions $\{w_p\}_{p=1}^{P}$ is chosen such that $\mathbf{G}$ is full rank, the linear system (5) has a unique solution. Similar system of equations is solved for each k to obtain all the elements of $\mathbf{A}^{-1}$. Hence we have the following [1]:

Theorem 1 *Let $\mathbf{A} \in GL_n(R)$. Assume $h, g \in M_{Aff}(R^n, R)$ such that $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$. Given measurements of h and g, then $\mathbf{A}$ can be uniquely determined if there exists a set of Lebesgue measurable functions $\{w_\ell\}_{\ell=1}^{p}$, $p \geq n$, such that the matrix $\mathbf{G}$ defined above, is full rank.*

In the following sections we show that the problem of obtaining an *explicit* solution for the parameters of an unknown sequence of affine transformations – whose direct solution requires a highly complex search in a function space – can be formulated as a *parameter estimation problem*. Moreover, it is shown that the original problem can be formulated in terms of an *equivalent* problem which is expressed in the form of a *linear* system of equations in the unknown parameters of the affine transformation sequence. A solution of this linear system of equations provides an explicit solution for the unknown transformation parameters.

3. THE ALGORITHMIC SOLUTION FOR A TIME DEPENDENT EVOLUTION

In this section we extend the derivation in [1], which is briefly summarized in Section 2, to the case where the affine transformation changes with time. The set of observations in this case is the time sequence $h(\mathbf{x}, t)$.

3.1. Sequential Instantaneous Estimation

Sequential instantaneous deformation estimation is based on the following: At each time instant t, the transformation relating the pose of the object at time t relative to that at time $t-1$ is estimated. This is implemented by a direct application of the algorithm summarized in Section 2, assuming that for every t, $\mathbf{A}(t) \in GL_n(R)$ and that $h(\mathbf{x}, t) \in M_{Aff}(R^n, R)$. Thus using the notation of the previous section we assign for each t, $h(\mathbf{x}) = h(\mathbf{x}, t)$ and $g(\mathbf{x}) = h(\mathbf{x}, t-1)$, and solve for $\mathbf{A}^{-1}$ at each time instant t. Clearly this estimation method employs only the information in the two consecutive time instants.

Let $\{e_i(t) : R \to R\}$ be a set of linearly independent functions in $L_2(R)$ (*e.g.*, polynomials, trigonometric functions). Assume that the time dependence of $\mathbf{A}^{-1}$ is parameterized as

$$\begin{aligned} \mathbf{A}^{-1}(t) &= \begin{pmatrix} \sum\limits_{i=1}^{L} a_{11}^i e_i(t) & \cdots & \sum\limits_{i=1}^{L} a_{1n}^i e_i(t) \\ \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{L} a_{n1}^i e_i(t) & \cdots & \sum\limits_{i=1}^{L} a_{nn}^i e_i(t) \end{pmatrix} \\ &= \sum_{i=1}^{L} \begin{pmatrix} a_{11}^i & \cdots & a_{1n}^i \\ \vdots & \ddots & \vdots \\ a_{n1}^i & \cdots & a_{nn}^i \end{pmatrix} e_i(t) \quad t \in R \quad (6) \end{aligned}$$

The problem then is to determine the above sequence of L matrices from the previously obtained sequence of estimates $\{\hat{\mathbf{A}}_t^{-1}\}$. We thus employ in a second stage a least-squares solution to (6) to obtain a global parametric model for $\hat{\mathbf{A}}^{-1}(t)$ based on the estimated sequence $\{\hat{\mathbf{A}}_t^{-1}\}$.

3.2. Global Estimation of the Time Evolution

In the framework of global estimation of the dependence of the deformation in time, it is assumed that the given template $g(\cdot)$ is fixed in time. The set of observations is the time sequence $h(\mathbf{x}, t)$. Thus, the problem statement is as follows: Let $\mathbf{A}(t) \in GL_n(R)$ for every t. Assume that $g \in M_{Aff}(R^n, R)$ and that for every t, $h(t) \in M_{Aff}(R^n, R)$ such that

$$h(\mathbf{x}, t) = g(\mathbf{A}(t)\mathbf{x}) \quad (7)$$

Thus, given $h(t)$ and g, find $\mathbf{A}(t)$, which means, as in the previous solution, that we need to determine the sequence of L matrices in (6).

Define the following n-dimensional product $e_{i_1}(t)e_{i_2}(t)\cdots e_{i_n}(t)$ where the indices $i_1, i_2, \dots, i_n \in \{1, \dots, L\}$ and let

$$\{e_{i_1}(t)e_{i_2}(t)\cdots e_{i_n}(t)\}_{i_1,i_2,\dots i_n=1}^{L}$$

be the set of all such products. Clearly there are at most L^n different elements in this set. Let Q be the number of linearly independent element in this set. To simplify the notation we shall refer to this set using the notation $\{f_k(t)\}_{k=1}^{Q}$. By definition, the determinant of $\mathbf{A}^{-1}(t)$ in (6) has the form

$$|\mathbf{A}^{-1}|(t) = \sum_{k=1}^{Q} d_k f_k(t) \tag{8}$$

Extending the previous derivations to the time varying case we have

$$\int_{R^n} w_p \circ h(\mathbf{x}, t) = \int_{R^n} w_p \circ g(\mathbf{A}(t)\mathbf{x}) = |\mathbf{A}^{-1}|(t) \int_{R^n} w_p \circ g(\mathbf{y}) \tag{9}$$

Substituting (8) into (9) we have

$$\int_{R^n} w_p \circ h(\mathbf{x}, t) = \sum_{k=1}^{Q} d_k f_k(t) \int_{R^n} w_p \circ g(\mathbf{y}) \tag{10}$$

Let $[t_1, \dots, t_v]^T$ denote the vector of time samples and let $\{w_i\}_{i=1}^{P}$ be the sequence of chosen left-compositions. Then,

$$\begin{pmatrix} \int_{R^n} w_1 \circ h(\mathbf{x}, t_1) \\ \vdots \\ \int_{R^n} w_1 \circ h(\mathbf{x}, t_v) \\ \vdots \\ \vdots \\ \int_{R^n} w_P \circ h(\mathbf{x}, t_v) \end{pmatrix} =$$

$$\begin{pmatrix} f_1(t_1) \int_{R^n} w_1 \circ g(\mathbf{y}) & \cdots & f_Q(t_1) \int_{R^n} w_1 \circ g(\mathbf{y}) \\ \vdots & & \vdots \\ f_1(t_v) \int_{R^n} w_1 \circ g(\mathbf{y}) & \cdots & f_Q(t_v) \int_{R^n} w_1 \circ g(\mathbf{y}) \\ \vdots & & \\ \vdots & & \\ f_1(t_v) \int_{R^n} w_P \circ g(\mathbf{y}) & \cdots & f_Q(t_v) \int_{R^n} w_P \circ g(\mathbf{y}) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_Q \end{pmatrix}$$

By choosing the $\{w_i\}_{i=1}^{P}$ such that at least Q rows of this matrix are linearly independent $|\mathbf{A}|(t)$ is found by a LS solution. Clearly, if the model is accurate only $P = Q$ equations are required.

Having obtained the Jacobian of the transformation we next determine the entries of $\mathbf{A}^{-1}(t)$ for each of its rows and for all $t \in \{t_1, \dots, t_v\}$.

$$\begin{aligned}
\int_{R^n} x_k w_p \circ h(\mathbf{x}, t) \quad &= \quad \int_{R^n} x_k w_p \circ g(\mathbf{A}(t)\mathbf{x}) \\
&= |\mathbf{A}^{-1}(t)| \int_{R^n} ((\mathbf{A}^{-1}(t)_k\, \mathbf{y}) w_p \circ g(\mathbf{y}) \\
&= |\mathbf{A}^{-1}(t)| \int_{R^n} \left(\sum_{j=1}^{n} (\sum_{i=1}^{L} a_{kj}^{i} e_i(t)) y_j \right) w_p \circ g(\mathbf{y}) \Big) \\
&= \sum_{j=1}^{n} \left(|\mathbf{A}^{-1}(t)| \sum_{i=1}^{L} a_{kj}^{i} e_i(t) \right) \int_{R^n} y_j w_p \circ g(\mathbf{y}) \Big) \\
&= \sum_{i=1}^{L} \sum_{j=1}^{n} a_{kj}^{i} \left(e_i(t) |\mathbf{A}^{-1}(t)| \int_{R^n} y_j w_p \circ g(\mathbf{y}) \right)
\end{aligned} \tag{11}$$

which yields when expressed in a matrix form

$$\begin{pmatrix} \int_{R^n} x_k w_1 \circ h(\mathbf{x}, t_1) \\ \int_{R^n} x_k w_1 \circ h(\mathbf{x}, t_2) \\ \vdots \\ \int_{R^n} x_k w_1 \circ h(\mathbf{x}, t_v) \\ \int_{R^n} x_k w_2 \circ h(\mathbf{x}, t_1) \\ \vdots \\ \int_{R^n} x_k w_P \circ h(\mathbf{x}, t_v) \end{pmatrix} = [\mathbf{G}_t^1, \mathbf{G}_t^2, \dots, \mathbf{G}_t^L] \begin{pmatrix} a_{k1}^1 \\ a_{k2}^1 \\ \vdots \\ a_{kn}^1 \\ a_{kn}^2 \\ \vdots \\ a_{kn}^L \end{pmatrix} \tag{12}$$

where

$$\mathbf{G}_t^i = [\mathbf{G}_t^{i,1}, \mathbf{G}_t^{i,2}, \dots, \mathbf{G}_t^{i,n}] \tag{13}$$

and

$$\mathbf{G}_t^{i,j} = \begin{pmatrix} e_i(t_1)|\mathbf{A}^{-1}|(t_1) \int_{R^n} y_j w_1 \circ g(\mathbf{y}) \\ e_i(t_2)|\mathbf{A}^{-1}|(t_2) \int_{R^n} y_j w_1 \circ g(\mathbf{y}) \\ \vdots \\ e_i(t_v)|\mathbf{A}^{-1}|(t_v) \int_{R^n} y_j w_P \circ g(\mathbf{y}) \end{pmatrix} \tag{14}$$

Hence,

Theorem 2 *Let $\mathbf{A}(t) \in GL_n(R)$ for every t. Assume that for every t, $h(t), g \in M_{Aff}(R^n, R)$ such that $h(\mathbf{x}) = g(\mathbf{A}(t)\mathbf{x})$. Then given measurements of $h(t)$ and g, $\mathbf{A}(t)$ can be uniquely determined if there exists a set of measurable functions $\{w_\ell\}_{\ell=1}^{P}$ such that the matrix*

$$[\mathbf{G}_t^1, \mathbf{G}_t^2, \dots, \mathbf{G}_t^L] \tag{15}$$

is full rank.

4. NUMERICAL EXAMPLE

The example illustrates the operation of the proposed algorithms on a car image shown in Figure 1. The image coordinate system is $[-1, 1] \times [-1, 1]$. A sequence of 200 affine deformations is applied to this image to create the observed time sequence. The car motion model is an approximation of a pure rotation by a third order Taylor series expansion. The evolution in time of the deformation model is depicted in Fig. 2 by the red curve, for each of the entries of the deformation matrix $\mathbf{A}^{-1}(t)$. The sequence of estimated transformations, $\{\mathbf{A}_t^{-1}\}$, obtained by applying the sequential instantaneous estimation procedure described in Section 3.1 is depicted by the blue curves. Obviously, since no global information about the continuity of the motion is used, the resulting estimates are noisy. However, when a global polynomial model is fit to this noisy sequence of estimates, a smooth curve which is very close to the trajectory of the deformation, is obtained. It is depicted in green. The results obtained by the global estimation procedure described in Section 3.2 are essentially identical to those obtained by fitting the same type of model (polynomial) to the sequence of sequential estimates $\{\mathbf{A}_t^{-1}\}$ (depicted by the green trajectory). It is therefore concluded that the sequential, pairwise, estimates of the time evolution of the deformation lead to an estimate of the time evolution for the entire observation period which is of the same accuracy as in the case where a global model is used, by employing a considerably simpler computational procedure.

Fig. 1. An observation from the car sequence.

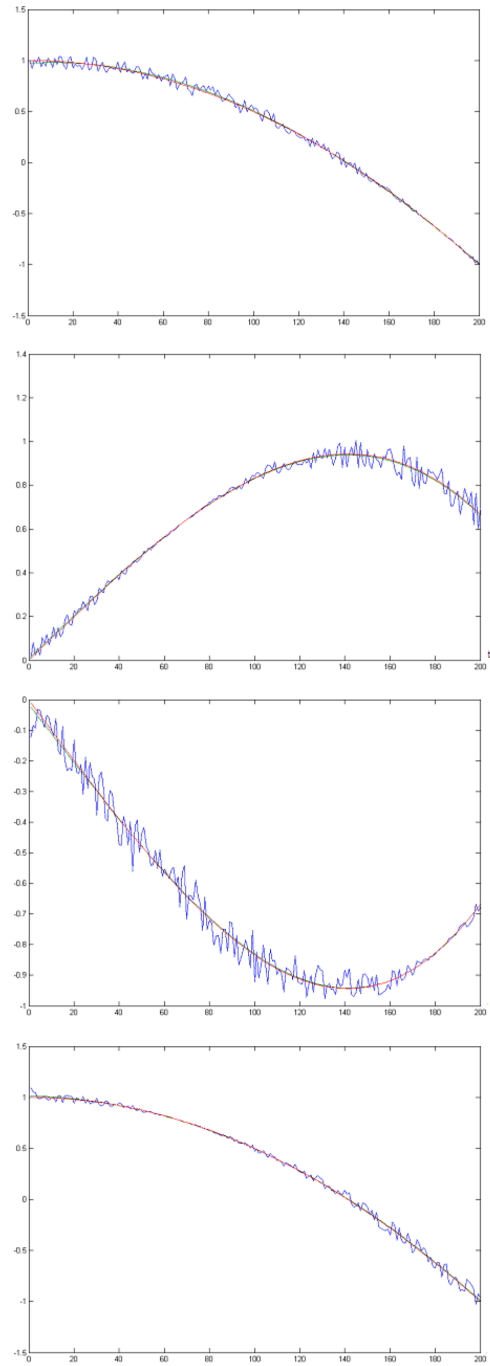

Fig. 2. Time evolution of the transformation matrix $\mathbf{A}^{-1}$. From top to bottom: $\mathbf{A}_{11}^{-1}, \mathbf{A}_{12}^{-1}, \mathbf{A}_{21}^{-1}, \mathbf{A}_{22}^{-1}$. Red: the true function of time of each entry; Blue: the sequential instantaneous estimate of each entry as a function of time; Green: the estimated evolution in time of each entry, both by the polynomial fit to the sequential estimate and the global one.

5. REFERENCES

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