TIGHTER PERFORMANCE BOUNDS ON IMAGE REGISTRATION

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ABSTRACT

Image registration is a fundamental and important task in image processing. It essentially estimates a transformation that aligns two images. Cramer Rao Lower bound has recently been used to establish the performance limit of image registration algorithms. However, it is known to be a weak lower bound for some problems. In this paper, we analyze the mean square error performance of transformation estimation in image registration problems We focus on rigid body transformations, and derive a set of tighter alternatives, namely the Bhattacharya bound and the Ziv-Zakai bound. Experimental results demonstate the validity of our performance bounds.

1. INTRODUCTION

The primary objective of image registration is to match two images that differ in certain aspects, e.g., translation, scaling and rotation (assuming a rigid-body misalignment), but essentially represent the same scene. A transformation is to be found so that the points and objects in one image can be related to their corresponding points and objects in the other images.

Performance analysis for image registration is usually performed visually. However, a visual inspection is not a satisfactory evaluation method and hence one needs to explore other evaluation techniques. Robinson and Milanfar [1] proposed that Mean Square Error be used as a standard performance measure to provide a fair comparison between different algorithms. Some results on metrics and bounds, e.g., the Cramer-Rao lower bound (CRLB), for image registration were obtained. Yetik and Nehorai [2] extended the results in [1] to the case of general transformations. But in their results, the rigid body transformation parameters are the functions of unknown rotation angle and are hard to obtain in practice.

CRLB is typically used as a benchmark for mean squared error performance in estimation problems. However, it is known to be an optimistic bound that may not be tight enough to provide meaningful insight into the achievable estimator performance. There are other lower bounds that may, for certain SNR regions, be better indicators of estimator performance. For this reason, this paper investigates the Bhattacharyya bound (BB) and the Ziv-Zakai bound (ZZB) for the estimation of the transformation in image registration.

The CRLB and the BB belong to the family of deterministic ”covariance inequality” bounds, which treat the parameter as an unknown deterministic quantity, and provide bounds on the Mean Square Error (MSE) in estimating any selected value of the parameter [3]. In the case of vector estimation problems, the CRLB and BB of one parameter are usually functions of other unknown parameters which makes the problem of computing the CRLB and BB difficult in practice. The Ziv-Zakai bound is a Bayesian bound, which assumes that the parameter is a random variable with known *a priori* distribution. It provides a bound on the global MSE averaged over the *a priori* probability density function [3]. Thus, we incorporate knowledge of the *a priori* transformation parameter space via the *a priori* distribution and predict the performance of image registration using this Bayesian bound. Our goal is to identify robust and suitable metrics and bounds on the performance of registration algorithms.

2. IMAGE REGISTRATION PROBLEM

Image registration is defined as a mapping between two images spatially so that they are aligned. The objective is to estimate the appropriate transformation parameters. Let $I_1(x, y)$ and $I_2(x, y)$ denote the two images to be registered. Each image can be considered as a noise-free image plus noise. The misalignment transformation is assumed to be applied to the noise-free image. Assuming $u(x, y)$ and $v(x, y)$ to be the transformed coordinates, the two images can be modeled as:

$$\begin{aligned} I_1(x,y) &= f(x,y) + n(x,y), \\ I_2(x,y) &= f(u(x,y),\, v(x,y)) + n(x,y). \end{aligned} \tag{1}$$

In this paper we assume the noise $n(x, y)$ to be i.i.d Gaussian noise with variance N. Thus each pixel is considered as a Gaussian random variable with the mean equal to the noise-free pixel intensity and variance of N.

3. RIGID BODY TRANSFORMATION

Misalignment error defined as a rigid body transformation has the following mapping function:

$$\begin{aligned} u &= \cos\theta_0 x + \sin\theta_0 y + x_0 \\ v &= -\sin\theta_0 x + \cos\theta_0 y + y_0 \end{aligned} \tag{2}$$

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The parameters to be estimated for registration are rotation θ_0, and translations x_0 and y_0.

3.1. Cramer Rao Lower bound

The CRLB for the iamge registration problem can be expressed as[4]:

$$CRLB\left(\hat{\theta}_i\right) = I^{-1}(i,i) \quad i = 1,2,3. \tag{3}$$

where I is the Fisher information matrix (FIM) with:

$$I_{ij} = E\left[\frac{\partial^2 LogP(I_1, I_2)}{\partial\theta_i \partial\theta_j}\right] \tag{4}$$

$P(I_1, I_2)$ is the joint probability distribution function of the images I_1 and I_2; $\theta = \{\theta_0, x_0, y_0\}$ is the set of parameters to be estimated. Let **r** denote the coordinates of the image (u, v). Then the FIM of the rigid body transformation can be expressed by:

$$\begin{aligned} I = \tfrac{1}{N}\sum_{\mathbf{r}} \Big\{ &\Big[\Big(f_u(\mathbf{r})\tfrac{du}{d\theta_0} + f_v(\mathbf{r})\tfrac{dv}{d\theta_0}\Big) \quad f_u(\mathbf{r}) \;\; f_v(\mathbf{r})\Big] \\ &* \Big[\Big(f_u(\mathbf{r})\tfrac{du}{d\theta_0} + f_v(\mathbf{r})\tfrac{dv}{d\theta_0}\Big) \quad f_u(\mathbf{r}) \;\; f_v(\mathbf{r})\Big]^T \Big\} \end{aligned} \tag{5}$$

where $f_u(\mathbf{r})$ and $f_v(\mathbf{r})$ are the derivatives of the image along the x axis and y axis respectively. Then the CRLB for a given θ_0 is:

$$CRLB\left(\hat{\theta}_i|\theta_0\right) = I^{-1}(i,i) \quad i = 1,2,3. \tag{6}$$

3.2. Bhattacharyya bound

The BB can be expressed as [5][6]:

$$BB\left(\hat{\theta}_i\right) = J_M^{-1}(i,i) \quad i = 1,2,3. \tag{7}$$

where J_M is the M^{th} order FIM with the size $3M * 3M$, which can be expressed as:

$$J_M = \begin{bmatrix} I_{11} \cdots I_{1M} \\ \vdots \quad \ddots \quad \vdots \\ I_{M1} \cdots I_{MM} \end{bmatrix} \tag{8}$$

The $(m, n)^{th}$ element of the $3 * 3$ matrix $I_{k,r}$ is defined as:

$$[I_{k,r}]_{m,n} = E_\theta\left\{\frac{\partial^k \log P(I_1, I_2)}{\partial\theta_m^k}\frac{\partial^r \log P(I_1, I_2)}{\partial\theta_n^r}\right\} \tag{9}$$

for $k, r = 1, ..., M$ and $m, n = 1, 2, 3$.

The element I_{11} is the FIM. So when $M = 1$, the BB reduces to the CRLB. Increasing M will mean more computational effort but is expected to yield a tighter bound. Here we consider the second order FIM for rigid body transformation which is given by:

$$J_2 = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \tag{10}$$

$$\begin{aligned} I_{11}(i,j) &= \tfrac{1}{N}\sum_{\mathbf{r}} \tfrac{\partial f(\mathbf{r})}{\partial\theta_i}\tfrac{\partial f(\mathbf{r})}{\partial\theta_j} \\ I_{12}(i,j) &= \tfrac{1}{N}\sum_{\mathbf{r}} \tfrac{\partial f(\mathbf{r})}{\partial\theta_i}\tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_j^2} \\ I_{21}(i,j) &= \tfrac{1}{N}\sum_{\mathbf{r}} \tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_i^2}\tfrac{\partial f(\mathbf{r})}{\partial\theta_j} \\ I_{22}(i,j) &= \tfrac{1}{N}\sum_{\mathbf{r}} \tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_i^2}\tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_j^2} + \tfrac{1}{N^2}\sum_{\mathbf{r}}\left(\tfrac{\partial f(\mathbf{r})}{\partial\theta_i}\right)^2 \sum_{\mathbf{r}}\left(\tfrac{\partial f(\mathbf{r})}{\partial\theta_j}\right)^2 \end{aligned} \tag{11}$$

In the second order FIM given in (11), for the rigid body transformation we have

$$\begin{aligned} &\tfrac{\partial f(\mathbf{r})}{\partial\theta_1} = f_u(\mathbf{r})\tfrac{du}{d\theta_0} + f_v(\mathbf{r})\tfrac{dv}{d\theta_0} \\ &\tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_1^2} = -f_u(\mathbf{r})\,u - f_v(\mathbf{r})\,v + f_{uu}(\mathbf{r})\left(\tfrac{du}{d\theta_0}\right)^2 \\ &\qquad + (f_{uv} + f_{vu})\tfrac{du}{d\theta_0}\tfrac{dv}{d\theta_0} + f_{vv}(\mathbf{r})\left(\tfrac{dv}{d\theta_0}\right)^2 \\ &\tfrac{\partial f(\mathbf{r})}{\partial\theta_2} = f_u(\mathbf{r}),\; \tfrac{\partial f(\mathbf{r})}{\partial\theta_3} = f_v(\mathbf{r}) \\ &\tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_2^2} = f_{uu}(\mathbf{r}),\; \tfrac{\partial^2 f(\mathbf{r})}{\partial\theta_3^2} = f_{vv}(\mathbf{r}) \end{aligned} \tag{12}$$

Thus, the BB is given by:

$$BB\left(\hat{\theta}_i|\theta_0\right) = J^{-1}(i,i) \quad i = 1,2,3. \tag{13}$$

In the above results, the CRLB and BB are functions of the rotation angle. If we assume that the rotation angle is uniformly distributed in $[0, \Delta_\theta]$, we can obtain the CRLB and BB as follows,

$$\begin{aligned} CRLB\left(\hat{\theta}\right) &= \int_0^{\Delta_\theta} CRLB\left(\hat{\theta}|\theta_0\right)\cdot P(\theta_0)\,d\theta_0 \\ BB\left(\hat{\theta}\right) &= \int_0^{\Delta_\theta} BB\left(\hat{\theta}|\theta_0\right)\cdot P(\theta_0)\,d\theta_0 \end{aligned} \tag{14}$$

3.3. Ziv Zakai Bound

While deriving CRLB and BB, we assumed that the intensity of each image pixel has a Gaussian distribution. We may also assume the transformation parameters, θ_0, x_0 and y_0 to be random variables. Here we assume that the rotation and translations x and y are all uniformly distributed random variables in $[0, \Delta_\theta]$, $[0, \Delta_x]$ and $[0, \Delta_y]$ respectively.

3.3.1. Translation error

Based on the result in [3], the ZZB for the translation error along x axis is:

$$ZZB(\hat{x_0}) = \int_0^{\Delta_x}\left\{\int_0^{\Delta_\theta} A(\theta_0, h)\,P_{\min}(\theta_0, h)\,d\theta_0\right\} h dh \tag{15}$$

where $A(\theta_0, h) = \min\{P(\theta_0, t), P(\theta_0, t + h)\}$; $P_{\min}(\theta_0, h)$ is the probability of error associated with the hypothesis testing problem

$$\begin{aligned} H_0 &: x_0 = t \\ H_1 &: x_0 = t + h \end{aligned} \tag{16}$$

where the two hypotheses are equally likely. Here t is the true translation and h is the translation error.

For the estimation error h, the probability of error P_{min} is given by:

$$P_{\min}(\theta_0, h) = Q\left(\sqrt{\frac{E(1-\rho(h))}{2N}}\right) \geq \begin{cases} Q\left(\frac{h}{2}\sqrt{\frac{r(c+\cos(2\theta_0+\delta))}{N}}\right), & 0 \leq h \leq \frac{\sqrt{2}}{\beta} \\ Q\left(\sqrt{\frac{E}{2N}}\right), & h > \frac{\sqrt{2}}{\beta} \end{cases} \tag{17}$$

where

$$\begin{aligned} &F(w,v) \text{ is the Fourier transform of the image } f(x,y); \\ &Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt; \quad E = \sum_{x,y} f^2(x,y); \\ &\rho(h) = \frac{1}{E}\sum_{x,y} f(x,y) f(x-h,y); \\ &M_1 = \int_w \int_v |F(w,v)|^2 w^2 dw dv; \\ &M_2 = \int_w \int_v |F(w,v)|^2 v^2 dw dv; \\ &M_3 = \int_w \int_v |F(w,v)|^2 wv dw dv; \\ &r = \sqrt{M_3^2 + (M_1 - M_2)^2/4}; \\ &\delta = \arcsin(M_3/r); \quad c = \frac{M_1+M_2}{2r}; \\ &\beta = \sqrt{r(c+\cos(2\theta_0+\delta))/E}. \end{aligned} \tag{18}$$

Using a property in [3],

$$\frac{1}{2}\int_0^{\Delta_x} \left\{ \int_0^{\Delta_\theta} A(\theta_0, h)\, d\theta_0 \right\} h dh = \frac{\Delta_x^2}{12} \tag{19}$$

and substituting Equation (17) into (15), yields

$$\begin{aligned} ZZB(\hat{x}_0) &= \frac{\Delta_x^2}{6} Q\left(\sqrt{\frac{E}{2N}}\right) + \int_0^{\Delta_\theta} \frac{1}{\Delta_\theta} \\ &\left\{ \int_0^{\frac{\sqrt{2}}{\beta}} \left[Q\left(\frac{h}{2}\sqrt{\frac{r(\cos(2\theta_0+\delta))}{N}}\right) - Q\left(\sqrt{\frac{E}{2N}}\right)\right] h dh \right\} d\theta_0 \\ &= \frac{\Delta_x^2}{6} Q\left(\sqrt{\frac{E}{2N}}\right) + \\ &\frac{N\Gamma_{1.5}\left(\frac{E}{4N}\right)}{\Delta_\theta r\sqrt{c^2-1}} \left[\tan^{-1}\left(\frac{\tan(\Delta_\theta+\delta/2)}{\sqrt{\frac{c+1}{c-1}}}\right) - \tan^{-1}\left(\frac{\tan(+\delta/2)}{\sqrt{\frac{c+1}{c-1}}}\right)\right] \end{aligned} \tag{20}$$

Similarly, we can obtain the translation error along y axis, which is:

$$\begin{aligned} ZZB(\hat{y_0}) &= \frac{\Delta_y^2}{6} Q\left(\sqrt{\frac{E}{2N}}\right) + \\ &\frac{N\Gamma_{1.5}\left(\frac{E}{4N}\right)}{\Delta_\theta r\sqrt{c^2-1}} \left[\tan^{-1}\left(\frac{\tan(\Delta_\theta+\delta/2)}{\sqrt{\frac{c-1}{c+1}}}\right) - \tan^{-1}\left(\frac{\tan(+\delta/2)}{\sqrt{\frac{c-1}{c+1}}}\right)\right] \end{aligned} \tag{21}$$

3.3.2. Rotation error

A Gaussian distribution in time domain is still a Gaussian distribution in Fourier domain due to the linear operation of Fourier transformation. Then a polar transform is made on the coordinates of the Fourier domain. It yields,

$$\begin{aligned} &F_1(\rho,\theta) = F(\rho,\theta) + n(\rho,\theta), \\ &F_2(\rho,\theta) = F(\rho,\theta-\theta_0)\, e^{-j\rho cos(\theta)x_0 - j\rho sin(\theta)y_0} + n(\rho,\theta). \end{aligned} \tag{22}$$

By applying the results in [3] here to obtain the ZZB on the rotation error, we have:

$$ZZB(\hat{\theta}_o) = \frac{\Delta_\theta^2}{6} Q\left(\sqrt{\frac{E}{2N}}\right) + \frac{N}{\beta_2^2 E}\Gamma_{\frac{3}{2}}\left(\frac{E}{4N}\right) \tag{23}$$

where

$$\begin{aligned} &E = \sum_{\rho,\theta} P^2(\rho,\theta) \\ &\beta_2^2 = \int_w \int_v |F_P(w,v)|^2 v^2 dw dv / \int_w \int_v |F_P(w,v)|^2 dw dv \end{aligned} \tag{24}$$

$P(\rho,\theta)$ is the polar transform of the Fourier transform of the image and $F_P(w,v)$ is the Fourier transform of $P(\rho,\theta_0)$.

4. EXPERIMENTAL RESULTS

For experimental verification, the image in Fig. 1 is used for simulations. We generated a pair of images with a random translation in x axis, translation in y axis and rotation uniformly distributed in $[0, 10]$, $[0, 10]$ and $[0, 10^o]$ respectively using Bi-cubic interpolation. Then white Gaussian noise was added to the image pair prior to registration. The FFT based Fourier method [7,8] and the Mutual Information (MI) based method [9,10] are used to estimate the translation and rotation errors. The whole process was repeated 500 times at each noise power value. Then we computed the root mean square error of the translation and rotation estimates and compared with CRLB, BB and ZZB. The results are shown in Fig.2 and Fig.3. In this experiment, BB and ZZB are observed to be tighter than CRLB. It is noticed that ZZB is 20dB higher than CRLB. Thus ZZB can characterize the performance more closely and can be used as a good bound for image registration problems.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we considered the problem of image registration performance evaluation from statistical estimation point of view. We focused our attention only on rigid transformations. CRLB is a standard performance bound, that is fairy simple to compute, but may be quite a loose bound. Further, in the case of rigid body transformation in image registration problems, we have three parameters to estimate, rotation, translation in x axis and translation in y axis. The CRLB of

Fig. 1. An Example Image

these parameters are functions of the unknown parameter rotation angle. This results in difficulty while obtaining a CRLB value in practice. This paper assumes some knowledge of the prior distribution of the transformation parameters and derives Bayesian bounds for translation and rotation errors in image registration problems. In addition, we find that image derivatives are involved in the calculation of these bounds, which implies that these bounds are affected by image content also. If an image contains higher frequencies, the image is more sensitive to image transformations, and therefore produces larger registration errors.

Our experimental results demonstrate that the ZZB is tighter than CRLB and BB for image registration problems. Here we assumed independence of image pixels. Future work will incorporate spatial correlation into the image model so as to obtain even tighter performance bounds in registration problems. In addition, performance bounds for other transformations such as affine transforms and perspective transforms, and performance bounds on multimodality images will be derived in the future.

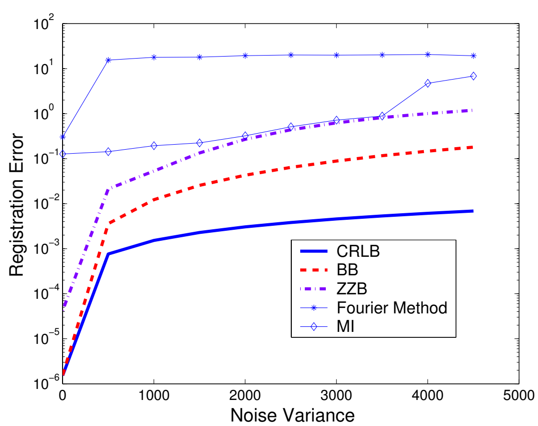

Fig. 2. Performance results for translation error

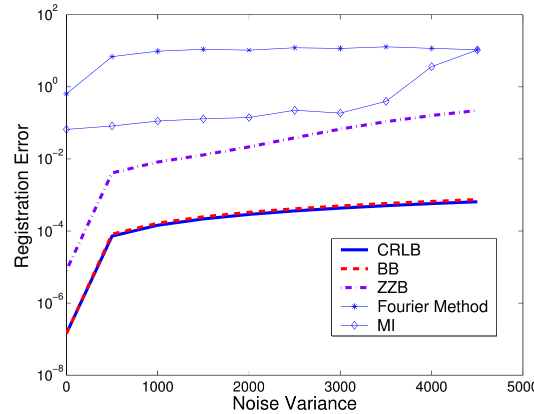

Fig. 3. Performance results for rotation error

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