SDP FOR 2-D FILTER DESIGN: GENERAL FORMULATION AND DIMENSION REDUCTION TECHNIQUES

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ABSTRACT

In this paper, a new technique for designing linear phase 2-D filter based on semi-definite programming (SDP) is proposed. This approach allows the design of 2-D filters with accurate cut-off frequency, subject to hard bounds on the frequency response to be achieved on a standard computer. Using the notion of 2-D trigonometric curves, we generalize the 2-D trigonometric Markov-Lukacs theorem to identify the pass-band and the stop-band in the region of support. The 2-D filter specifications are expressed as linear matrix inequalities. We also exploit convex duality to derive SDP formulations of reduced dimensions. Numerical examples illustrating the advantages of our method are also presented.

1. INTRODUCTION

Two-dimensional (2-D) digital filters and filter banks have found applications in many different fields and are the subject of intensive research (see e.g. [4, 8, 15] and the references therein). Despite the large variety of 2-D filter design techniques [8], 1-D filter based design methods in which the desired 2-D filters are either transformed from 1-D filters [8, 10] or separable into 1-D filters [3], are still the most popular.

A salient 1-D filter based approach is the McClellan transform method which maps the frequency points of a 1-D FIR filter into the frequency contours of the desired 2-D filter [10, 11]. This approach does not guarantee a small deviation from the desired 2-D filter in the support regions, nor an accurate cut-off frequency. In addition, the designed 2-D FIR filters are good only if the cut-off frequency is much less than π and the sharp of both pass-band and stop-band regions are adequately described by the level sets of just unique 2D filter of low dimension [8]. Although an improvement has been made to design wide-band FIR filter in [5], it is still very hard to find a 1-D prototype filter which satisfies the specifications incorporated from the 2-D stop-band and pass-band requirements. There are also extensions of heuristic griding techniques of 1-D filter to 2-D filters [2].

The semi-infinite constraints arising in 1-D filter design can be effectively addressed by semi-definite programming (SDP) base on the concept of 1-D trigonometric curves [14]. Although an SDP formulation of the 2-D positive real constraints has been developed in [6], the size of the SDP formulation is so large that it can only be applied to the design of very small 2-D filters.

In this paper, we propose a new approach to the design of 2-D filters. Our approach uses the concept of 2-D trigonometric curves with new bases introduced and then exploit convex duality to transform the design problem into an SDP of reasonably moderate dimension. Consequently, desirable 2-D filter can be designed using standard SDP solvers. In addition, our approach is able to achieve accurate cut-off frequency for each support region, and fully meet all other required specifications.

The paper is organized as follows. In Section 2 we give an explicit (semi-infinite) optimization formulation for the 2-D filter design problem. Then, by using a result of algebraic geometry [12] we show in Section 3, in a very general setting, that the semi-infinite trigonometric constraints can be addressed by SDP. Dimension reduction techniques for SDP is developed in Section 4 using a new approach. The problem dimension is further developed in Section 5 using convex duality. Numerical results to verify the viability of our result is presented Section 6 and concluding remarks are given in Section 7.

The notations used in the paper are rather standard, except $\langle A\rangle$ refers to the trace of a square matrix A, so $\langle AB\rangle = \langle BA\rangle$ for any matrices A and B of an appropriate size. By $X \geq 0$ ($X > 0$) we mean a symmetric positive (strictly positive) definite matrix X. One of the main condition of positive definite matrices that will be frequently used in the paper is $\langle XY\rangle \geq 0$ whenever $X \geq 0$ and $Y \geq 0$.

2. 2-D FILTER DESIGN

In this paper we consider the the linear phase four-fold filter $\mathcal{H}(z_1, z_2)$ which has frequency response

$$H(\Omega) = \sum_{i=0}^{n}\sum_{\ell=0}^{n} a_{i\ell}\cos(i\omega_1)\cos(\ell\omega_2) = \langle AM(\Omega)\rangle \quad (1)$$
$$= \varphi_n^T(\omega_1)A\varphi_n(\omega_2) \quad (2)$$

where

$$\Omega = (\omega_1, \omega_2);\ A = [a_{i\ell}]_{i,\ell=0}^n;\ M(\Omega) = \varphi_n(\omega_1)\varphi_n^T(\omega_2) \quad (3)$$
$$\varphi_n(\omega_i) = (1, \cos\omega_i, \cos 2\omega_i, \ldots, \cos n\omega_i)^T \quad (4)$$

The design of the filter $\mathcal{H}(z_1, z_2)$ involves the selection of the matrix of filter coefficients A such that the frequency response $H(\Omega)$ satisfy a given set of specifications. These specifications include:
(i) Minimal weighted-square error

$$W_p\int_{\Omega_p}|H(\Omega)-1|^2 d\Omega + W_s\int_{\Omega_S}|H(\Omega)|^2 d\Omega \quad (5)$$

where $\Omega_p = [0, \omega_{1p}]\times[0,\omega_{2p}]$, $\Omega_s = \Omega_{s1}\cup\Omega_{s2}, \Omega_{1s} = [\omega_{1s}, \pi]\times[0,\pi], \Omega_{2s} = [0,\omega_{1s}]\times[\omega_{2s},\pi]$ are pass-band and stop-band of $\mathcal{H}(z_1, z_2)$ respectively, while $d\Omega = d\omega_1 d\omega_2$.
(ii) Pass-band and stop-band peak-errors stay below the respective tolerances $\delta_p > 0, \delta_s > 0$, i.e. the semi-infinite constraint

$$|H(\Omega) - 1| < \delta_p\ \forall(\Omega) \in \Omega_p \quad (6)$$
$$|H(\Omega)| < \delta_s\ \forall(\Omega) \in \Omega_s \quad (7)$$

With simple calculation the above 2-D filter design problem can be rewritten as a minimization of a convex quadratic objective function over semi-infinite constraints

$$\min_A W_p\langle M_{1p}AM_{2p}A^T\rangle + W_s\langle M_{1s1}AM_{2s1}A^T\rangle$$
$$+W_s\langle M_{1s2}AM_{2s2}A^T\rangle - 2W_p\langle M_pA\rangle \quad \text{s.t.} \quad (8)$$
$$-\delta_p \le \langle A, M(\Omega)\rangle - 1 \le \delta_p \quad \forall(\Omega)\in\Omega_p \quad (9)$$
$$-\delta_s \le \langle A, M(\Omega)\rangle \le \delta_s \quad \forall\Omega\in\Omega_s = \Omega_{s1}\cup\Omega_{s2} \quad (10)$$

where $M_{ip} = \int_0^{\omega_{ip}}\varphi_n(\omega_i)\varphi_n(\omega_i)^T d\omega_i > 0$,
$M_{isi} = \int_{\omega_{is}}^{\pi}\varphi_n(\omega_i)\varphi_n(\omega_i)^T d\omega_i > 0, i = 1, 2;$
$M_{2s1} = \int_0^{\pi}\varphi_n(\omega_2)\varphi_n(\omega_2)^T d\omega_2 > 0,$
$M_{1s2} = \int_0^{\omega_{1s}}\varphi_n(\omega_1)\varphi_n(\omega_1)^T d\omega_1 > 0,$
$M_p = \int_0^{\omega_{1p}}\varphi_n(\omega_1)d\omega_1 \times \int_0^{\omega_{2p}}\varphi_n(\omega_2)d\omega_2.$

The 2-D semi-infinite constraints (9)-(10) poses the foremost difficulty in the design of 2-D filters. The simplest method is to constrain the peak pass-band and stop-band errors on a set of finite grid points in Ω_p and Ω_s. However, there is no guarantee that the constraints (9)-(10) are completely satisfied.

3. SDP AS FUNDAMENTAL TOOL HANDLING SEMI-INFINITE CONSTRAINTS

The above semi-infinite trigonometric constraints (9)-(10) are particular cases of the general semi-infinite constraint

$$g_0(\Omega) \ge 0\ \forall\Omega \in \{\Omega\in[0,\pi]^2 :\ g_i(\Omega) \ge 0,\ i = 1, 2, ..., m\}, \quad (11)$$

where $g_i(\Omega), k = 0, 1, ..., m$ are trigonometric polynomials in Ω, i.e.

$$g_i(\Omega) = \sum_\alpha g_{i\alpha}\cos\Omega^\alpha,\ \cos\Omega^\alpha = \cos(\alpha_1\omega_1)\cos(\alpha_2\omega_2). \quad (12)$$

To develop semi-definite characterizations of the trigonometric semi-infinite constraints, we introduce the trigonometric moment matrices

$$M_i(\Omega) = \phi_i(\Omega)\phi_i^T(\Omega), i = 1, 2, ... \quad (13)$$

where

$$\phi_i(\Omega) = (1, \cos\omega_1, \cos\omega_2, \cos 2\omega_1, \cos\omega_1\cos\omega_2, \cos 2\omega_2, \cos 3\omega_1, \ldots, \cos i\omega_2)^T \quad (14)$$

Clearly, all moment matrices $M_i(\omega)$ are positive semi-definite. We can prove the following result based on a result of the algebraic geometry [12].

Theorem 1 *The semi-infinite constraint (11) is fulfilled if and only if $g_0(\Omega)$ admits the following representation*

$$g_0(\Omega) = \langle X_0M_{r_0}(\Omega)\rangle + \sum_{i=1}^m g_i(\Omega)\langle X_iM_{r_i}(\Omega)\rangle \quad (15)$$

with some r_0, r_i and $X_0 \ge 0, X_i \ge 0$ of appropriate dimensions.

By comparing terms with the same trigonometric powers $\cos\Omega^\alpha$ at the both sides of (15), one can easily obtain the linear relationship between the coefficients of g_0 and entries of matrices X_0, X_i, which together with the constraints $X_0 \ge 0, X_i \ge 0$ constitute a set of semi-definite constraints that is equivalent to (11).
From a practical view point, the following issues are pertinent to the computational implementation of Theorem 1:
(i) Generally, the numbers r_0, r_i are not known in advance and they are potentially high;
(ii) The dimensions of the matrices M_{r_0}, M_{r_i} i.e. $(r_0+1)^2\times(r_0+1)^2$ and $(r_i+1)^2\times(r_i+1)^2$ increase very quickly as r_0, r_i increase. Consequently, the dimensions of the matrix variables X_0, X_i, which are the same as those of M_{r_0} and M_{r_i}, increase so quickly that the dimension of resultant SDP grows beyond the capacity of current SDP solvers.
To provide the reader with some perspectives on our developments in the subsequent sections, we discuss the dimensionality of the potential SDP arising from four 2-D trigonometric semi-infinite constraints in (9)-(10), each of them is a particular of (11) with some g_0 of order n and some affine (1-st order) $g_i, i = 1, 2, 3, 4$. For simplicity, consider n odd, i.e. $n = 2k+1$. Based on Theorem 1, the simplest sufficient condition for each trigonometric semi-infinite constraint in (9)-(10) is (15) that with the minimal $r_i = k$ chosen beforehand to make the highest power on the right side of (15) match that of the left hand size. Accordingly, the dimension of X_i is $(k+1)^2\times(k+1)^2$. Thus, the total number of scalar variables in the resultant SDP is $5(k+1)^2((k+1)^2+1)/2$ which is already in the order of several thousands for a very modest $n = 15$.

4. DIMENSION REDUCTION WITH FLEXIBLE BASES

In the 1-D case, the numbers r_0, r_i in (15) are exactly determined from orders of g_i via the trigonometric Markov-Lukacs theorem [14]. The starting point of this section is the following adaptation of Theorem 1.

Theorem 2 *(**Generalized 2-D trigonometric Markov-Lukacs theorem**) Suppose that g_0 is a 2-D trigonometric polynomial of order n*

$$g_0(\Omega) = \sum_{i=0}^{n}\sum_{\ell=0}^{n} g_{i\ell}\cos(i\omega_1)\cos(\ell\omega_2)$$

Then $g_0(\Omega) \geq 0 \; \forall \cos\Omega \in [\cos a, \cos b] \times [\cos c, \cos d]$ if

$$\begin{array}{rcl} g_0(\Omega) & = & (\cos\omega_1 - \cos a)(\cos\omega_2 - \cos c)\langle X_1\Psi_1(\Omega)\rangle \\ & & +(\cos b - \cos\omega_1)(\cos d - \cos\omega_2)\langle X_2\Psi_2(\Omega)\rangle \\ & & +(\cos\omega_1 - \cos a)(\cos d - \cos\omega_2)\langle X_3\Psi_3(\Omega)\rangle \\ & & +(\cos\omega_2 - \cos c)(\cos b - \cos\omega_1)\langle X_4\Psi_4(\Omega)\rangle \end{array} \tag{16}$$

for some $X_i \geq 0$ and $\Psi_i(\Omega) \geq 0 \; \forall\Omega \in [0,\pi]^2$, $i = 1,2,3,4$.

Note that we don't impose a structure like (13) of moment matrices to $\Psi_i(\Omega)$ in (16). A novel and effective construction for them is the first objective in this section.
Note that

$$\begin{array}{l} (\cos\omega_1 - \cos a)(\cos\omega_2 - \cos c) = T_{11}(\Omega) - a_1 \\ (\cos b - \cos\omega_1)(\cos d - \cos\omega_2) = T_{21}(\Omega) - a_2 \\ (\cos\omega_1 - \cos a)(\cos d - \cos\omega_2) = T_{31}(\Omega) - a_3 \\ (\cos\omega_2 - \cos c)(\cos b - \cos\omega_1) = T_{41}(\Omega) - a_4 \end{array} \tag{17}$$

where

$$\begin{array}{l} T_{11}(\Omega) = -\cos c\cos\omega_1 - \cos a\cos\omega_2 + \cos\omega_1\cos\omega_2 \\ T_{21}(\Omega) = -\cos d\cos\omega_1 - \cos b\cos\omega_2 + \cos\omega_1\cos\omega_2; \\ T_{31}(\Omega) = \cos d\cos\omega_1 + \cos a\cos\omega_2 - \cos\omega_1\cos\omega_2; \\ T_{41}(\Omega) = \cos c\cos\omega_1 + \cos b\cos\omega_2 - \cos\omega_1\cos\omega_2; \\ a_1 = -\cos a\cos c;\; a_2 = -\cos b\cos d; \\ a_3 = \cos a\cos d;\; a_4 = \cos c\cos b \end{array} \tag{18}$$

For $j = 1,2,3,4$, the following definition is consistent with (18)

$$\begin{array}{r} T_{j0}(\Omega) \equiv 1; \\ T_{ji}(\Omega) = 2T_{j(i-1)}(\Omega)T_{j1}(\Omega) - T_{j(i-2)}(\Omega),\; i = 2,3,..., \end{array} \tag{19}$$

Lemma 1 *For any i, ℓ the following relation holds true*

$$T_{ji}(\Omega)T_{j\ell}(\Omega) = \frac{1}{2}(T_{j(i+\ell)}(\Omega) + T_{j|i-\ell|}(\Omega)) \tag{20}$$

Next, define the following base

$$\Psi_j(\Omega) = \begin{bmatrix} 1 \\ T_{j1}(\Omega) \\ .... \\ T_{jk}(\Omega) \end{bmatrix} \begin{bmatrix} 1 \\ T_{j1}(\Omega) \\ .... \\ T_{jk}(\Omega) \end{bmatrix}^T \tag{21}$$

The following proposition is a direct consequence of Theorem 2.

Corollary 1 *For $p = (a,b,c,d)$, let $\mathcal{C}_p$ be the cone defined by*

$$\begin{array}{l} \mathcal{C}_p = \{X \in R^{(2k+2)\times(2k+2)} : \\ \langle X, M(\Omega)\rangle \equiv \displaystyle\sum_{i=1}^{4}\langle X_i, (T_{i1}(\Omega) - a_i)\Psi_i(\Omega)\rangle, X_i \geq 0\}, \end{array} \tag{22}$$

with $T_{j1}(\Omega)$ and a_i given by (18).
If $X \in \mathcal{C}_p$ then

$$\langle XM(\Omega)\rangle \geq 0 \quad \forall\cos\Omega \in [\cos a, \cos b] \times [\cos c, \cos d] \tag{23}$$

The above result states that the cone $\mathcal{C}_p$ is a subset of the $n-$order 2-D trigonometric polynomials that are nonnegative on $[\cos a, \cos b] \times [\cos c, \cos d]$.
In comparison with the dimension $(k+1)^2 \times (k+1)^2$ of the variables X_i in (15) for the simplest case that we mentioned at the end of the previous section, the dimension of X_i in (22) is $(k+1) \times (k+1)$, i.e. the substantial dimension reduction has been achieved with the new representation (22).
Before closing this section, we remark that our approach is also directly applicable to diamond-shaped constraint [1], fan-shaped constraint, and more sophisticated shaped constraints involving elliptic or parabolic ones.

5. DUAL SDP FOR FURTHER DIMENSION REDUCTION

We now turn our attention back to the 2-D filter design problem that requires minimizing the quadratic objective (8) over the 2-D trigonometric semi-infinite constraints (9)-(10). The crucial step is to show that the dual cone $\mathcal{C}_p^*$ of $\mathcal{C}_p$ (defined by (22)) is described by the following SDP of moderate dimension

$$\mathcal{C}_p^* = \{Y \in R^{(2k+2)\times(2k+2)} : \Theta_i(Y) \geq 0,\; i = 1,2,3,4\} \tag{24}$$

where Y and $\Theta_i(Y)$ are obtained from $M(\Omega)$ and $(T_{i1}(\Omega) - a_i)\Psi_i(\Omega)$ through the variable change

$$\cos\Omega^\alpha \rightarrow y_\alpha = y_{\alpha_1\alpha_2} \quad \forall\alpha \tag{25}$$

i.e. $\cos\ell\omega_1 \rightarrow y_{\ell 0},\; \cos\ell\omega_2 \rightarrow y_{0\ell},\; \cos i\omega_1\cos\ell\omega_2 \rightarrow y_{i\ell}$.
Now, based on the Corollary 1, we effectively strengthen the semi-infinite constraints (9)-(10) by the semi-definite constraints

$$\begin{array}{c} A - (1-\delta_p)E_1 \in C_p,\; -A + (1+\delta_p)E_1 \in C_p \\ A + \delta_s E_1 \in C_{s_i},\; -A + \delta_s E_1 \in C_{s_i},\; i = 1,2 \end{array} \tag{26}$$

where $E_1 \in R^{(2k+2)\times(2k+2)}$ with $E_1(0,0) = 1$ and $E_1(i,\ell) = 0$ for $i + \ell > 0$ and $p = (\omega_{1p}, 0, \omega_{2p}, 0)$ while $s_1 = (\pi, \omega_{1s}, \pi, 0)$, $s_2 = (\omega_{1s}, 0, \pi, \omega_{s2})$.
The optimization problem (8), (26) thus belongs to the following class of problems

$$\begin{array}{rl} \displaystyle\min_X & \displaystyle\sum_{i=1}^{L}\langle M_{1i}XM_{2i}X^T\rangle - \langle MX\rangle \\ \text{s.t} & A_iX + D_i \in \mathcal{C}_i,\; i = 1,2,\cdots,m, \end{array} \tag{27}$$

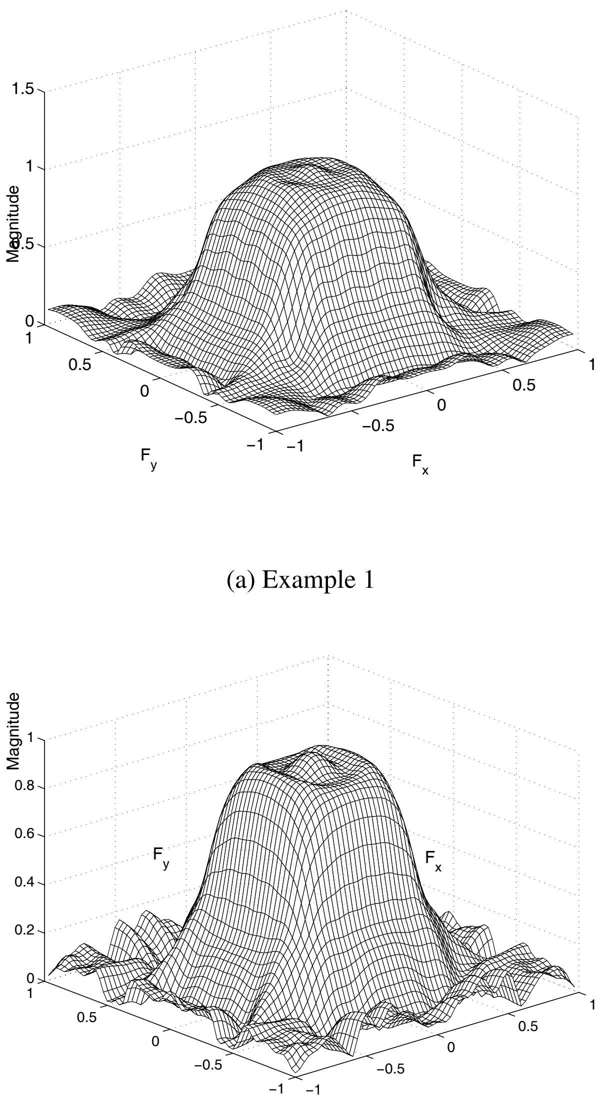
where $\mathcal{C}_i$ are cones like $\mathcal{C}_p$ defined by (22) and thus their dual cones $\mathcal{C}_i^*$ are completely described by semi-definite constraints like (24), $M_{1i} > 0, M_{2i} > 0$ and M are given matrices.
Then, using the Lagrange multiplier approach, the dual of SDP (27) can be shown as

$$\begin{array}{r} \displaystyle\max_{Y^{(i)},X_{opt}} [-\sum_{i=1}^{m}\langle Y^{(i)}D_i\rangle - \sum_{i=1}^{L}\langle M_{1i}X_{opt}M_{2i}X_{opt}^T\rangle] : Y^{(i)} \in C_i^*, \\ 2\displaystyle\sum_{i=1}^{L} M_{2i}X_{opt}^T M_{1i} - (M + \sum_{i=1}^{m} Y^{(i)}A_i) = 0, \end{array} \tag{28}$$

which is an SDP because the objective is a convex quadratic functional in $Y^{(i)}, X_{opt}$ and the cones $\mathcal{C}_i^*$ are described by the SDP like (24). Solving the SDP (28) yields the optimal solutions to the primal and dual optimization problems (27) and (24).

6. SIMULATION

This section presents numerical results obtained using the techniques described above. These simulations were performed using SeDuMi [13] and the its interface [9] in a MATLAB environment.
Two design examples of 2-D FIR filter are illustrated. In the first example, the desired frequency response of the four-fold 2D-FIR filter was a 19×19 square as shown in Fig 1(a). The specifications for the filter were $\omega_{1p} = \omega_{2p} = 0.4\pi, \omega_{1s} = \omega_{2s} = 0.6\pi, \delta_p = \delta_s = 0.1$.



(a) Example 1

(b) Example 2

Fig. 1. Frequency response of four-fold 2D-FIR filters.

In example 2, we consider a 2-D filter with narrow transition band ($\omega_{1p} = \omega_{2p} = 0.4\pi, \omega_{1s} = \omega_{2s} = 0.5\pi, \delta_p = \delta_s = 0.1$) and bigger size ($27 \times 27$).
Figures 1(b) show the frequency response of the 2D-FIR filter for the second example. As can be seen, there is a steep decrease between pass-band and stop-band. The numerical results of the matrix A in these two examples can be provided if requested.

7. CONCLUSION

We have presented a new technique for designing linear phase 2-D filter based on moderate SDP. The technique is very flexible and is capable of achieving accurate cut-off frequency and other practical sharp specifications for $2-D$ filters of practical length.

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