ANALYTIC FUNCTIONS, SINGULARITIES AND EDGES: A NEW FORMALISM

Mohamed Elfataoui and Gagan Mirchandani

Department of Electrical and Computer Engineering
The University of Vermont
Burlington, Vermont 05405, USA

ABSTRACT

Methods for detecting edges, be they multidimensional or multiresolution, ultimately reduce to finding extremal points, first derivatives or zeros of second derivatives. However, problems such as missing edges, weak edges due to thresholding, derivatives not existing and false edge generation, are some of the consequences. We adopt a new formalism: Edges are singularities of the mathematically smoothest function possible - the complex analytic function. We embed a real image into the real part of an analytic function. After solving the conjugate harmonic problem, edges in discrete images are identified from the imaginary part. The analytic function model is inherently two-dimensional and an invariant measure. Comparisons are made with other standard edge detection methods. We outline issues that need to be considered for establishing analytic functions for edge detection.

1. INTRODUCTION

Identifying edges in one-dimensional signals invariably entails as a first process, a search for local extrema of the smoothed (continuous) signals. Procedures for finding extremal points can be divided into four groups. These, with some typical examples are, gradient based methods [7], [1], methods based on the orientation of the gradient [3], those based on the phase of the Fourier component [4], [2] and finally, methods based on fitting a model [5].

There are problems with these approaches. They are inherently one-dimensional methods. Extension to higher dimensions using tensor products and the corresponding cartesian sampling structure, is not efficient [6] leading to poor representation of discontinuous functions. Even in the one-dimensional case, there can be problems identifying extremal points: The first is that in many applications, the values at the edge points may vary, as for example in the Canny edge detection [1] method, thus possibly loosing weak edges. Taking derivatives can also be problematic. Search for extrema itself can generate false edges.

In this work we introduce a new formalism for defining edges. While edges are typically identified through extremal points, edges can also arise through discontinuities of various orders and other irregularities of functions. To converge on an appropriate definition of an edge, we ask the question: What is not an edge? A first response may be C^k functions, that is functions having k continuous derivatives where k is a positive integer. Function $f(x) = |x|^3$ has two continuous derivatives at $x = 0$ but not three. The function has a "corner" at $x = 0$. Hence it is not an edge if $k < 3$, but an edge otherwise. In general, we may wish to use C^∞, that is, infinitely differentiable functions called *smooth* functions as the constraint for non-edges. But there are still C^∞ functions, such as the *bump* function [8] which looks like a bump (edge), and hence does not satisfy our goal of "complete" smoothness for non-edges.

Are there smooth functions that do not include the *bump* function? The answer is "yes" - *analytic functions*. The bump function is not an analytic function (ibid.) Accordingly, we choose the latter function to formalize our definition of a *non-edge*. An analytic function is the smoothest function possible. Hence, we will define an *edge* as that which corresponds to points where the function is not analytic. All other definitions of edges - extrema, discontinuities, irregularities, etc., are subsumed by this definition.

The paper is organized as follows: In Section 2, we discuss singularities, analytic functions, show images of analytic functions and define edges. In Section 3 we solve the conjugate harmonic problem and show how we use the imaginary part modulus to find edges in discrete images. Experimental results and comparison with other methods are shown in Section 4. In Section 5 we describe issues that need to be considered for establishing the analytic function model for edge detection.

1-4244-0469-X/06/$20.00 ©2006 IEEE

2. SINGULARITIES, ANALYTICAL FUNCTIONS AND EDGES

Intuitively, a singularity is a point where a function, equation, surface etc., misbehaves or becomes degenerate. In the mathematical theory of singularity, a singularity is a point at which a given mathematical object is not defined or where it fails to be well-behaved. In analytic function theory, singularities are points where the function *fails* to be analytic. That is, it is not a C^{∞} function which is complex differentiable.

Analytical functions have singularities, classified as removable, pole or essential. In our new definition, it is the pole and essential singularities that constitute edge points. A complex function $f(z)$, z complex, is called *analytic* when it is complex differentiable in an open domain in the z-plane. The necessary conditions for a function $f(z) = u(z) + jv(z)$ to be differentiable at a point are called the Cauchy-Riemann equations:

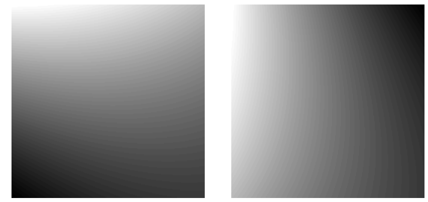
$$\frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y} \qquad \frac{\partial u(z)}{\partial y} = -\frac{\partial v(z)}{\partial x}. \tag{1}$$

An analytic function has derivatives of all orders in the region in which it is analytic. When $f(z)$ is analytic, both the real and imaginary parts $u(z), v(z)$ are harmonic functions, that is, they satisy Laplace's equation.

As an illustration of analytic functions and their smoothness property, we consider two simple examples. Let $f_1(x, y) = y^3 - 3x^2y + i(x^3 - 3xy^2)$, and $f_2(x, y) = 2x(1 - y) + i(x^2 - y^2 + 2y)$. Figures (1) and (2) show the real and imaginary part of the analytic function f_1 and f_2 respectively. We observe that *both* two elements of the analytic function - real and imaginary parts, are smooth functions that are either monotonically increasing or decreasing in some direction. None of them can have discontinuities of any order; none of them can have a modulus that has a local maximum. Essentially, all the maps are devoid of any and all significant features. These characteristics are typical of analytic functions. Any deviation from this infinite smoothness results in non-analyticity.

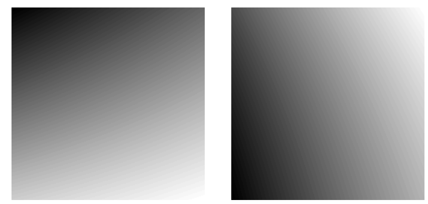
Definition: An *edge* of an image $u(x, y)$ is the set of points where the associated complex function has a pole or essential singularity. Equivalently, where its conjugate harmonic function $v(x, y)$ has infinite magnitude or does not exit.

The application of this definition of an edge to the edge detection problem in discrete images, proceeds as follows. A continuous function $u(x, y)$ is derived



(a) (b)

Figure 1: (a) Real part, (b) Imaginary part.



(a) (b)

Figure 2: (a) Real part, (b) Imaginary part.

from the original discrete image $u(m, n)$ by interpolation or approximation. This is assumed to form the real part of an analytic function $f(x, y)$, that is $f(x, y) = u(x, y)+jv(x, y)$. Solving the conjugate harmonic problem using the Cauchy Riemann equations, generates the imaginary part $v(x, y)$. Points (singularities) where the signal is not analytic, that is, where edges exist, are identified by points where $|v(x, y)|$ is infinite. This follows from the assumption that $|u(x, y)|$, stemming as it does from a bounded $u(m, n)$, is bounded. Correspondingly, and in practice, we obtain that $|v(m, n)|$ is very large, even for low magnitude singularities.

We observe two features of the analytic function model: The first is that it is an invariant measure in that any deviation from infinite smoothness, no matter how small, leads to an infinite $|v(x, y)|$. Secondly, imposing an analytic constraint on the image inherently imposes a 2-dimensional (non-separable) characterization.

3. THE CONJUGATE HARMONIC PROBLEM

We have the following problem, known in the literature as the conjugate harmonic problem: For $f(x, y) = u(x, y) + jv(x, y)$ an analytic function in some domain D, given $u(x, y)$ can we find $v(x, y)$? We proceed as follows: We interpolate $u(x, y)$ of the discrete image $u(p, q)$ in such a way that allows the constant in $v(x, y)$ to be a function of only x. We achieve this by (i)utilizing a tensor product of cardinal B-splines and (ii) through selection of the order of the B-splines. Finally, use of a separable basis makes the analysis tenable. We determine the interpolated function $u(x,y)$

$$u(x,y) = \sum_{n=1,m=1}^{N,M} a_{m,n} B_m(x) B_n(y).$$

such that

$$u(x,y)|_{(x,y)=(p,q)} = u(p,q),$$

where $u(p,q)$ is the given data.

We use the first Cauchy-Riemann condition to obtain $v(x,y)$ from $u(x,y)$ as

$$\begin{aligned} v(x,y) &= \int_1^y \frac{\partial u(x,v)}{\partial x} dv + C(x) \\ &= \sum_{n=1,m=1}^{N,M} a_{m,n} \int_1^y B'_m(x) B_n(v) dv + C(x) \\ &= \sum_{n=1,m=1}^{N,M} a_{m,n} B'_m(x) \int_1^y B_n(v) dv \\ &\quad + C(x), \end{aligned} \tag{2}$$

where C is a function that depends only of the variable x. We will see how to choose the order of the B-spline to satisfy this condition. Then, applying the second of the two Cauchy Riemann conditions to equation (2), we get

$$\begin{aligned} C'(x) &= \frac{\partial v(x,y)}{\partial x} - \\ &\quad \frac{\partial(\sum_{n=1,m=1}^{N,M} a_{m,n} B'_m(x) \int_1^y B_n(v)dv)}{\partial x} \\ &= -\frac{\partial u(x,y)}{\partial y} - \\ &\quad \frac{\partial(\sum_{n=1,m=1}^{N,M} a_{m,n} B'_m(x) \int_1^y B_n(v)dv)}{\partial x} \\ &= -\sum_{n=1,m=1}^{N,M} a_{m,n} B_m(x) B'_n(y) - \\ &\quad \frac{\partial(\sum_{n=1,m=1}^{N,M} a_{m,n} B'_m(x) \int_1^y B_n(v)dv)}{\partial x} \\ &= -\sum_{n=1,m=1}^{N,M} a_{m,n} B_m(x) B'_n(y) - \\ &\quad \sum_{n=1,m=1}^{N,M} a_{m,n} B''_m(x) \int_1^y B_n(v) dv. \end{aligned} \tag{3}$$

In order for the function C to depend only on x, the terms depending on y in equation (3) must be eliminated. Therefore, taking the order of the tensor B-spline equal to 2, i.e; $m = 2$ and $n = 2$, we get

$$B'_n(y) = a_n, \quad \text{where } a_n \text{ are constants,}$$

and

$$B''_m(x) = 0.$$

Therefore

$$\sum_{n=1,m=1}^{N,M} a_{m,n} B''_m(x) \int_1^y B_n(v) dv = 0$$

Hence

$$C'(x) = -\sum_{n=1,m=1}^{N,M} a_n a_{m,n} B_m(x) \tag{4}$$

Therefore C will not depends on y. Thus from equation (4), we get

$$\begin{aligned} C(x) &= -\int_1^M \sum_{n=1,m=1}^{N,M} a_n a_{m,n} B_m(u) du \\ &= -\sum_{n=1,m=1}^{N,M} a_n a_{m,n} \int_1^x B_m(u) du + b, \end{aligned} \tag{5}$$

where b is a constant. We choose $b = 0$. Therefore, combining equations (2) and (5) we get

$$\begin{aligned} v(x,y) &= \sum_{n=1,m=1}^{N,M} a_{m,n} B'_m(x) \int_1^y B_n(v) dv - \\ &\quad \sum_{n=1,m=1}^{N,M} a_n a_{m,n} \int_1^x B_m(u) du, \end{aligned} \tag{6}$$

Therefore

$$\begin{aligned} v(p,q) &= \sum_{n=1,m=1}^{N,M} a_{m,n} B'_m(p) \int_1^q B_n(v) dv - \\ &\quad \sum_{n=1,m=1}^{N,M} a_n a_{m,n} \int_1^p B_m(u) du. \end{aligned} \tag{7}$$

4. EXPERIMENTAL RESULT

Given a discrete image $u(p,q)$ we generate the imaginary part $v(p,q)$ of $f(p,q) = u(p,q) + iv(p,q)$ using equation (7). Input image data has integer values between 0 and 255. $|v(p,q)|$ is mapped to unsigned 8-bit integers. The reason is that image features have very large values. We use an upper bound of 255 to make the detection invariant to the image intensity. In addition, since we are using the tensor product as a scheme for interpolation, we apply the same algorithm to four flipped versions of the image and then flip back the result. The four results are added.The algorithm is applied locally using a 5×5 window.

Results from the new method are compared empirically with those from two known methods [1], [2]. We do not include non-maximal suppression or threshold processing. For the Canny edge detector, we use $\sigma = 1$. We first note that the background of both test images, Figures (3-(a), (e)) look like harmonic functions without any interesting features. As expected, in the new method Figures (3-(b), (f)), we do not see any features in the background. In contrast, the Phase Congruency method, Figures (3-(c), (g)) exhibit artifacts. Edges in the new method are clean and have almost the same magnitude in contrast to Canny's method, Figures (3-(d), (h)) where the magnitude of the edges vary.

5. FURTHER INVESTIGATION

The quality of the analytic function based edge detector needs to be determined and performance evaluated with respect to localization and spurious response. Different interpolation and approximation schemes in two-dimensions, such as triangulation, moving least squares scheme and using multiquadrics need to be considered.

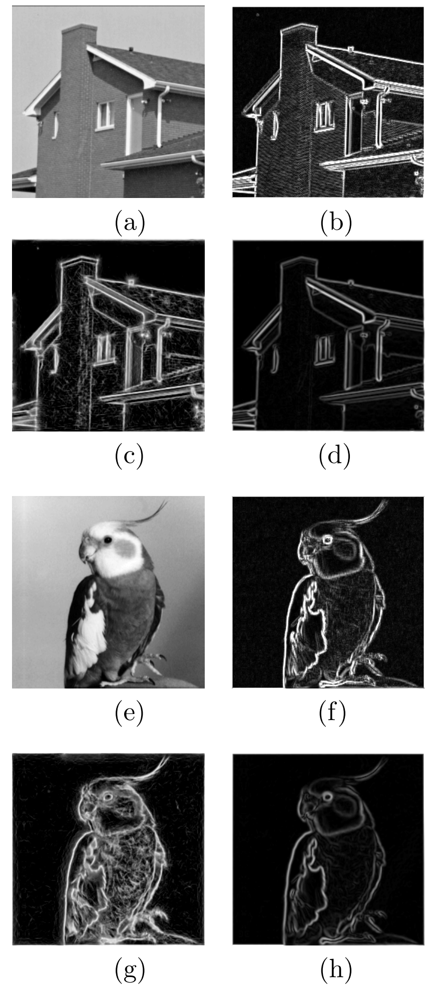


Figure 3: (a),(e) Test images, (b),(f) New method, (c),(g) Phase Congruency, (d),(h) Canny.

6. REFERENCES

[1] J.F. Canny, "A Computational Approach to Edge Detection," *IEEE Trans. PAMI*, vol. 8, no. 6, pp. 679-698, 1986.

[2] P. Kovesi. *Invariant measure of image features from phase information.* Ph.D thesis, University of western Australia, May 1996.

[3] R. Machuca and A.L. Gilbert, "Finding Edges in Noisy Scenes," *IEEE Trans. PAMI*, vol. 3, no. 1, pp. 103-111, 1981.

[4] M. C. Morrone and R. Owens, "Feature detection from local energy," *Pattern Recognition Letters*, Vol. 6, pp. 303-313, 1987.

[5] V.S. Nalwa and T.O. Binford, "On Detecting Edges," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 8, no. 6, pp. 699-714, 1986.

[6] D.P.Peterson and D. Middleton, "Sampling and Reconstruction of Wave-Number-Limited Functions in N-Dimensional Euclidean Spaces," *Information and Control*, 5(4), December 1962, pp.279-323.

[7] J. M. S. Prewitt, *Object enhancement and extraction in Picture Processing and Psychopictorics*, Academic Press, 1970.

[8] Eric W.Weisstein et al.,"C-Infty Function," http://mathworld.wolfram.com/C-InfinityFunction.html.