QUANTIZATION ON THE GRASSMANN MANIFOLD: APPLICATIONS TO PRECODED MIMO WIRELESS SYSTEMS

Bishwarup Mondal, Robert W. Heath Jr. [*]

The University of Texas at Austin
{mondal,rheath}@ece.utexas.edu

Leif W. Hanlen [†]

National ICT Australia
leif.hanlen@nicta.com.au

ABSTRACT

This paper studies the problem of quantization of a source that lives on the complex Grassmann manifold. The special structure of the Grassmann manifold and the distortion measures that are defined on it differentiates this problem from the traditional problem of vector quantization in Euclidean spaces. Assuming a uniform source distribution along with a distortion based on chordal distance, codebook design algorithms are mentioned and rate distortion tradeoffs are studied. The expected distortion for such a quantizer is approximately characterized. These results are then applied to the performance analysis of a multiple antenna wireless communication system.

1. INTRODUCTION

The Grassmann manifold has been explored by the signal processing community in diverse contexts - optimization, estimation and channel coding among others. The problem of quantization on the Grassmann manifold, on the other hand, has received less attention. Recently, quantization problems on the Grassmann manifold have generated interest, partly due to applications in multiple-input multiple-output (MIMO) wireless communication. The design and analysis of quantizers on the Grassmann manifold, which exploit the geometrical structure of the manifold, is the topic of this present paper.

The geometrical exposition of the Grassmann manifold presented here is from the viewpoint of computational linear algebra (see e.g. [1]). Let U_n be the unitary group defined as the set of n-by-n unitary matrices. Let $V_{n,p}$ be the Stiefel manifold that is the set of all p-dimensional bases in an n-dimensional space. The Grassmann manifold $G_{n,p}$ is a quotient manifold and may be defined as the set of all p-dimensional subspaces of an n-dimensional space. An element of $G_{n,p}$ can be identified by those bases in $V_{n,p}$ that span the same subspace. An n-by-p orthonormal matrix can represent a p-dimensional basis in n-space. Thus, points in the Grassmann manifold are equivalence classes of n-by-p orthonormal matrices (two matrices are equivalent if their columns span the same subspace). In quotient notation, $G_{n,p} = V_{n,p}/U_p$.

A point in $G_{n,p}$ is a p-dimensional linear subspace and may be represented by an arbitrary basis in the form of an n-by-p orthonormal matrix. The matrix representation of a point in $G_{n,p}$ is non-unique and $Y \in G_{n,p}$ (Y orthonormal) essentially means that the column space of Y is an element of $G_{n,p}$.

Let us consider a random matrix source Y, where Y is n-by-p and orthonormal. We use the same notation for the random matrix as well as a realization of it. Considering Y as an element of the Euclidean space $\mathbb{C}^{n\times p}$, we can define a quantizer $\mathcal{Q}$ of size N as a map [1]

$$\mathcal{Q} : \mathbb{C}^{n\times p} \mapsto \mathcal{W},\ \text{where } \mathcal{W} = \{Y_1, \cdots, Y_N\},\ Y_i \in \mathbb{C}^{n\times p}. \quad (1)$$

The set $\mathcal{W}$ is a codebook of size N and each element is a codeword or a reproduction point. The resolution or rate of the quantizer is given by $\log_2 N/(2np)$. This is a traditional definition of a quantizer in the Euclidean space [2]. To measure the performance of the quantizer $\mathcal{Q}$, we define distortion d as a map

$$d : \mathbb{C}^{n\times p} \times \mathbb{C}^{n\times p} \mapsto \mathbb{R}^+ \cup \{0\} \quad (2)$$

and a performance metric for the quantizer is the expected distortion given by $E\{d(Y, \mathcal{Q}(Y))\}$, where the expectation is over the domain of definition of the source Y.

Now let us consider the case when (i) $\mathcal{W} = \{Y_1, Y_2 \cdots, Y_N\}$, Y_i orthonormal, and (ii) $d(Y, \mathcal{Q}(Y)) = d(YQ_1, \mathcal{Q}(Y)Q_2)$ for any $Q_1, Q_2 \in U_p$. Then the quantization problem in (1) can be formulated in $G_{n,p}$ as

$$\mathcal{Q} : G_{n,p} \mapsto \mathcal{W},\ \text{where } \mathcal{W} = \{Y_1, \cdots, Y_N\},\ Y_i \in G_{n,p}. \quad (3)$$

In this case the rate of the quantizer is not well defined, since either $2np$ or the dimension of $G_{n,p}$ may be used to normalize $\log_2 N$. Consequently, we study the performance of the quantizer (3) as a function of N.

Formulating the quantization problem on $G_{n,p}$ (as opposed to $\mathbb{C}^{n\times p}$) enables us to exploit the redundancy arising from the geometry of manifold in the form of the:

(i) orthonormality constraint - the columns of Y are orthonormal,

(ii) rotation invariance - post multiplication of Y by any element of U_p results in zero distortion.

The objective of this paper is to discuss codebook design algorithms and study the rate-distortion function for a quantizer on the Grassmann manifold. As an example application, a MIMO maximum ratio transmission/combining scheme with quantized channel information is analyzed.

[*]Bishwarup Mondal is the recipient of a Motorola Partnerships in Research Grant. This material is based in part upon work supported by the Texas Advanced Tech. Program under Grant No. 003658-0380-2003.

[†]National ICT Australia is funded through the Australian Government's *Backing Australia's Ability initiative*, in part through the Australian Research Council.

[1]We use ^{H} to denote conjugate transposition, $\|\cdot\|_2$ to denote the matrix l_2-norm, $|\cdot|$ to denote absolute value, $\mathbb{C}^m$ to denote the m-dimensional complex vector space, $\mathcal{CN}(0, 1)$ to denote complex normal distribution with independent real and imaginary parts distributed according to $\mathcal{N}(0, 1/2)$

Signal processing techniques on the Grassmann manifold include optimization algorithms [1, 3–5], space-time code design [6, 7] and quantized precoded MIMO systems [8–11]. A nice exposition of the geometry of the Grassmann manifold is provided in [3] and [1] in the context of optimization and in [12, 13] in the context of codes or "packings". The algorithm presented for codebook design in the case of $G_{n,1}$ is the Lloyd algorithm for vector quantization with a $\sin\theta$ distance [2, 8, 14]. Another algorithm mentioned for codebook design in the general case of $G_{n,p}$ is presented in [6]. The rate distortion analysis for $G_{n,1}$ is also presented partly in [14] in the context of performance analysis of MIMO beamforming combining systems. In the case of $G_{n,p}$, an asymptotic analysis for rate distortion is proposed [12, 15].

In Section 2, the quantization problem on $G_{n,p}$ is formally stated and distortion functions are defined, and codebook design algorithms are presented. Rate-distortion analysis is presented in Section 3. This theory is applied to a MIMO beamforming scheme in Section 4.

2. PROBLEM DESCRIPTION

Let us consider a source that lives in the complex Grassmann manifold $G_{n,p}$. A probability measure can be defined over $G_{n,p}$ using the Haar measure. In this paper, we characterize the performance of the quantizer assuming that the probability distribution of the source is uniform (isotropic). Then

$$f(Y) = k(n,p), \quad Y \in G_{n,p} \tag{4}$$

where $k(n,p)$ is a constant derived from the volume of $G_{n,p}$ and normalizes the density function [12, 16]. The notion of distortion in $G_{n,p}$ can be defined based on the principal angles between the subspaces $Y_1, Y_2 \in G_{n,p}$. Let $\{\theta_i\}_{i=1}^{p}$ be the principal angles between the subspaces spanned by the columns of Y_1 and Y_2. This also means that the singular value decomposition of $Y_1^H Y_2$ is $U(\cos\Theta)V^H$ where $(\cos\Theta) = \text{diag}(\cos\theta_1, \cdots, \cos\theta_p)$. Assume that the distortion function in $G_{n,p}$ is the chordal distortion defined as [2]

$$d(Y_1, Y_2) = \frac{1}{2}\|Y_1Y_1^H - Y_2Y_2^H\|_2^2 = \|\sin\Theta\|_2^2. \tag{5}$$

One of the advantages of the chordal distortion is the fact that with the metric of chordal distance (the positive square root of chordal distortion) $G_{n,p}$ can be isometrically embedded into a sphere of radius $\sqrt{p(n-p)/n}$ in $\mathbb{R}^{n^2-1}$ [12, 13]. Consequently, bounds on the size of spherical codes can be applied to codebooks in $G_{n,p}$.

Let $\mathcal{W} = \{W_i\}_{i=1}^N$, $W_i \in G_{n,p}$ be the codebook for quantization. We restrict ourselves to nearest neighbor quantizers. The Voronoi regions corresponding to each codeword induced by the quantizer is then given by (assuming that ties are resolved arbitrarily)

$$V(W_i) = \{Y \in G_{n,p} : d(Y, W_i) \le d(Y, W_j), j \ne i\}. \tag{6}$$

The expected distortion of the quantizer as a function of the cardinality of the codebook can be written as

$$D(N) = E\{d(Y, \mathcal{Q}(Y))\} = \sum_{i=1}^{N} \int_{V(W_i)} d(Y, W_i) f(Y) dY. \tag{7}$$

[2]This definition of chordal distortion is the square of the chordal distance defined in [13]. Note: from [1, 12] $\sqrt{d(Y_1,Y_2)}$ is a metric, but $d(Y_1, Y_2)$ is not necessarily a metric.

A distortion optimal codebook for a given value of N is given by

$$\mathcal{W}^* = \arg\min_{\mathcal{W}} D(N) \tag{8}$$

and the function $D(N)$ is termed the distortion rate function of $\mathcal{W}$. **Remark on Codebook Design:** It is intuitive that the codebook design criterion in (8) aims to choose N points that are uniformly positioned in $G_{n,p}$. In the case of $G_{n,1}$, the Lloyd algorithm for codebook design for vector quantization can be employed with the chordal distance metric to obtain a local optimum for the optimization problem in (8) (for details of the algorithm see [2, 8, 14]). For the general case of $G_{n,p}$, the optimization problem in (8) is difficult to solve directly and alternate cost functions have been proposed that capture this notion of uniformity. In this paper we consider the following max-min criterion for codebook design, defining an optimal codebook as

$$\mathcal{W}^\dagger = \arg\max_{\mathcal{W}} \Delta(\mathcal{W}) \tag{9}$$

$$\Delta(\mathcal{W}) = \min_{W_i, W_j \in \mathcal{W}, i \ne j} d(W_i, W_j). \tag{10}$$

It is observed that the Fourier based construction in [6] can be employed to generate codebooks for $G_{n,p}$ using the max-min criterion that provide good distortion performance.

3. DISTORTION RATE CHARACTERIZATION

Following (9), (10), let us define $\Delta^\dagger(N) \equiv \Delta(\mathcal{W}^\dagger)$, observing that $\Delta^\dagger(N)$ does not depend on any particular codebook. Let us also define a ball of radius γ centered at Y in $G_{n,p}$ as

$$B_\gamma(Y \in G_{n,p}) = \{X \in G_{n,p} : d(X, Y) \le \gamma\}. \tag{11}$$

The expected distortion $D(N)$ from (7) can be characterized as

$$\begin{aligned}
D(N) &= \sum_{i=1}^{N} \int_{V(W_i)} d(Y, W_i) f(Y) dY && (12)\\
&\ge \sum_{i=1}^{N} \int_{B_{\Delta(\mathcal{W})/2}(W_i)} d(Y, W_i) f(Y) dY && (13)\\
&= \sum_{i=1}^{N} \int_{G_{n,p}} P(B_{\Delta(\mathcal{W})/2}(W_i)) d(Y, W_i) \\
&\qquad \cdot f(Y | Y \in B_{\Delta(\mathcal{W})/2}(W_i)) dY && (14)\\
&= \sum_{i=1}^{N} P(B_{\Delta(\mathcal{W})/2}(W_i)) \\
&\qquad \cdot E\left\{d(Y, W_i) | Y \in B_{\Delta(\mathcal{W})/2}(W_i)\right\} && (15)\\
&= E\left\{d(Y, W_1) | Y \in B_{\Delta(\mathcal{W})/2}(W_1)\right\} \eta_{packing} && (16)
\end{aligned}$$

where $\eta_{packing}$ in (16) is the packing density defined as $\eta_{packing} = \sum_{i=1}^{N} \text{Vol}B_{\Delta(\mathcal{W})/2}(W_i)/\text{Vol}(G_{n,p})$. Note that $\eta_{packing}$ can be related to the packing density of spherical codes using the isotropic embedding of $G_{n,p}$ into $\mathbb{R}^{n^2-1}$ [13, 17]. The bound in (13) is due to the fact that $B_{\Delta(\mathcal{W})/2}(W_i) \subset V(W_i), \forall i$, (14) follows by defining $P(B_{\Delta(\mathcal{W})/2}(W_i)) = \int_{B_{\Delta(\mathcal{W})/2}(W_i)} f(Y)dY$, (16) follows since $P(B_{\Delta(\mathcal{W})/2}(W_i)) = \text{Vol}B_{\Delta(\mathcal{W})/2}(W_i)/\text{Vol}G_{n,p}$ and that the conditional expected distortion is independent of W_i due to the isotropic distribution.

The following lemma provides a way of characterizing the distortion.

Theorem 1. *If $\Delta^\dagger(N) > \Delta^\dagger(N+1)$, it follows that $G_{n,p} \subset \cup_{i=1}^N B_{\Delta^\dagger(N)}(W_i)$ where $W_i \in \mathcal{W}^\dagger$ and $|\mathcal{W}^\dagger| = N$. In other words, $\Delta^\dagger(N)$ is a covering radius for the code $\mathcal{W}^\dagger$.*

Proof: See [18]. □

The property, $\Delta^\dagger(N) > \Delta^\dagger(N+1)$ is satisfied for most cases of putatively optimal packings for practical values of N (see [13]). In the following discussion we assume that the cardinality of the codebook, N possesses this property. Then, from Theorem 1 and following a similar derivation as in (16) the distortion function $D(N)$ can be bounded as

$$\begin{aligned} E\left\{d(Y,W_1)|Y \in B_{\Delta^\dagger(N)/2}(W_1)\right\}\eta_{packing} \leq D^\dagger(N) \\ \leq E\left\{d(Y,W_1)|Y \in B_{\Delta^\dagger(N)}(W_1)\right\}\eta_{covering} \end{aligned} \quad (17)$$

where $\eta_{covering}$ is the covering density defined as $\eta_{covering} = \mathrm{Vol} \cup_{i=1}^N B_{\Delta^\dagger(N)}(W_i)/\mathrm{Vol}(G_{n,p})$. Thus, (17) characterizes the distortion $D(N)$ with lower and upper bounds depending on the packing density and covering density of the codebook $\mathcal{W}^\dagger$. Realizing that the optimal codebook $\mathcal{W}^\dagger$ is the best "packing", $\eta_{packing}$ is maximized by $\mathcal{W}^\dagger$.

In the particular case of $G_{n,1}$, the conditional expected distortion in a ball can be accurately characterized and the result is summarized in the following theorem.

Theorem 2. *In $G_{n,1}$, the chordal distortion in the ball $B_\gamma(Y \in G_{n,1})$ is given by*

$$E\{d(X,Y)|X \in B_\gamma(Y \in G_{n,1})\} = \frac{n-1}{n}\gamma^2. \quad (18)$$

Proof: See [18]. □

In the case of $G_{n,p}$, $p > 1$, an accurate characterization of conditional distortion is difficult. In the asymptotic regime, however, the distortion is characterized by the following.

Theorem 3. *In $G_{n,p}$, $p > 1$, with the chordal distance $d(Y_1,Y_2) = \|\sin\Theta\|_2$, as $n \to \infty$ the distortion in $B_\gamma(Y)$ is characterized by*

$$E\{d(X,Y)|X \in B_\gamma(Y \in G_{n,p})\} \geq \left[\frac{(n-2p)+1}{(n-2p)}\right]^{-(1+\frac{p}{2})}\gamma^2. \quad (19)$$

Proof: See [18]. □

The expected distortion $D^\dagger(N)$ can be characterized using (17) along with Theorems 2 and 3. In general, an accurate formula for $\Delta^\dagger(N)$ is not available but $\Delta^\dagger(N)$ can be characterized by bounds. In the case of $G_{n,1}$ a bound derived in [9, 10] can be expressed as

$$\Delta^\dagger(N) \leq 2\left(\frac{1}{N}\right)^{\frac{1}{2(n-1)}}, \quad |\mathcal{W}| = N. \quad (20)$$

In the general case of $G_{n,p}$, the Rankin bound (obtained by the isotropic embedding proposed in [13]) and a bound obtained from the Gilbert-Varshamov and Hamming inequalities of sphere packing using the volume of metric balls computed in [12] can be combined as

$$\Delta^\dagger(N) \leq \min\left\{p, \sqrt{\frac{p(n-p)N}{n(N-1)}}, \sqrt{2p\left(1-\left(1-\frac{1}{N^{1/np}}\right)\right)}\right\}. \quad (21)$$

It is observed that the Rankin bound is not tight at high values of N and the derivation of the other bounds as in (20), (21) subsumes that $\eta_{packing} = 1$. Thus, in the case of $G_{n,1}$ with large N, an approximation of $D^\dagger(N)$ can be derived using the bound in (20), the bound in (17) and Theorem 2 and the fact $\eta_{packing} = 1$ which gives

$$D^\dagger(N) \approx \left(\frac{n-1}{n}\right)\left(\frac{1}{N}\right)^{\frac{1}{n-1}}. \quad (22)$$

In the case of $G_{n,p}$, $p > 1$, using Theorem 3 and the fact that $\eta_{covering} > 1$, we can obtain an approximation of $D^\dagger(N)$ using the upper bound as

$$\begin{aligned} E\left\{d(Y,W_1)|Y \in B_{\Delta^\dagger(N)}(W_1)\right\}\eta_{covering} \\ > \left[\frac{(n-2p)+1}{(n-2p)}\right]^{-(1+\frac{p}{2})}\left(\Delta^\dagger(N)\right)^2 \end{aligned} \quad (23)$$

where $\Delta^\dagger(N)$ is estimated from (21).

In Table 1, the simulated distortion performance of practically designed codebooks (not necessarily $\mathcal{W}^\dagger$) is compared to the approximation of $D^\dagger(N)$ (as in (22)) in the case of $G_{4,1}$. The approximation of $D^\dagger(N)$ is close to the simulated distortion rate performance. In the case of $G_{10,2}$ Table 2 illustrates that the simulated distortion rate curve is close to the approximation obtained from (23).

4. EXAMPLE APPLICATION OF GRASSMANN QUANTIZATION

In this section, we provide an example application of quantization on $G_{n,1}$. Let us consider a narrowband MIMO wireless system with quantized beamforming and receive combining. Details of the system description and the problem can be found in [8–10]. Let the cardinality of the codebook $\mathcal{W}^\dagger$ be N, the number of transmit antennas be M_t and for simplicity let us consider a single receive antenna [3]. Define $\Gamma(N,\mathcal{W}^\dagger)$ as the expected loss in the signal-to-noise ratio of the system due to quantization. This is a measure of the difference between the channel gain due to transmission on the maximum eigenmode and the resultant channel gain due to transmission on the best choice of the codeword. Then,

$$\Gamma(N,\mathcal{W}^\dagger) = E\{\min_{W_i \in \mathcal{W}}(\lambda - \|HW_i\|^2)\}, \quad (24)$$

$$= E\{\lambda\}\left(1 - E\left\{\max_{1\leq i\leq N}|W_i^H Y|^2\right\}\right), \; Y \in G_{M_t,1} \quad (25)$$

$$= E\{\lambda\}E\left\{\min_{1\leq i\leq N} d^2(Y,W_i)\right\} \quad (26)$$

$$= E\{\lambda\}D^\dagger(N) \quad (27)$$

$$\approx E\{\lambda\}\frac{M_t-1}{M_t}\left(\frac{1}{N}\right)^{\frac{1}{M_t-1}} \quad (28)$$

$$= (M_t-1)\left(\frac{1}{N}\right)^{\frac{1}{M_t-1}} \quad (29)$$

where H is a $1 \times M_t$ vector with $[H]_i \sim \mathcal{CN}(0,1)$, λ is the eigenvalue of $H^H H$ and Y is the corresponding normalized eigenvector; (25) follows from the fact that λ and Y are independent, (26)

[3] The analysis presented here also extends to multiple receive antennas [14].

follows from the definition of chordal distance in (5), (27) follows from the definition of distortion rate function in (7) and the fact that Y is isotropic, (28) follows from (22) and (29) follows from the fact that $E\{\lambda\} = M_t$. In the particular case of $M_t = 3$, $M_r = 1$, (29) is evaluated and compared against the simulated loss in SNR, $\Gamma(N, \mathcal{W})$, for codebooks designed using the Lloyd algorithm mentioned in section 2. The results are plotted in Fig. 4 which shows that the lower bound provides a good estimate of the loss in SNR for practical values of N.

Table 1. Distortion-Rate for $G_{4,1}$.

N	Approx. of $D^{\dagger}(N)$	Simulated D(N)
2^2	0.4725	0.4800
2^3	0.3750	0.3929
2^4	0.2976	0.3090
2^5	0.2362	0.2507
2^6	0.1875	0.2005

Table 2. Distortion-Rate for $G_{10,2}$.

N	Simulated $D(N)$	Approx. of $D^{\dagger}(N)$
2^2	1.4049	1.5673
2^3	1.3338	1.3434
2^4	1.2743	1.2539
2^5	1.2210	1.2134
2^6	1.1707	1.1942

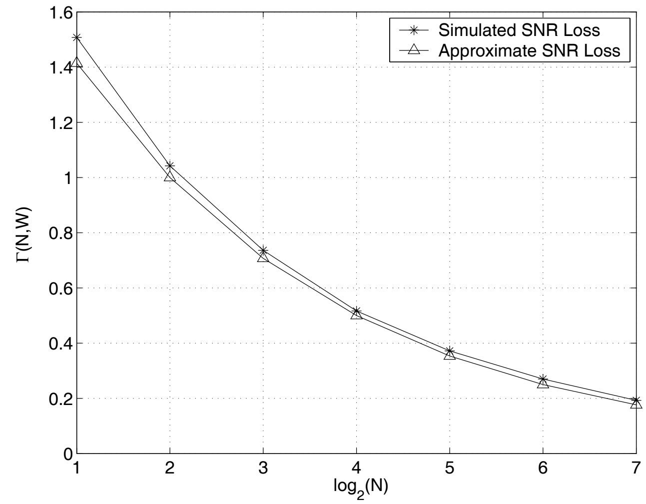

Fig. 1. The loss in SNR, $\Gamma(N, \mathcal{W})$, for $M_t = 3$, $M_r = 1$ is evaluated using simulations and plotted as a function of $\log_2 N$. The approximation of $\Gamma(N, \mathcal{W}^{\dagger})$ as in (29) is also plotted for comparison.

5. REFERENCES

[1] A. Edelman, T. A. Arias, and S. T. Smith, "The geometry of algorithms with orthogonality constraints," *SIAM J. Matrix Anal. Appl.*, vol. 20, no. 2, pp. 303 – 353, 1998.

[2] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*, Kluwer Academic Press, 1992.

[3] J. H. Manton, "Optimization algorithms exploiting unitary constraints," *IEEE Trans. on Sig. Proc.*, vol. 50, no. 3, pp. 635–650, Mar. 2002.

[4] J. H. Manton, "Modified steepest descent and Newton algorithms for orthogonally constrained optimisation. part II. the complex Grassmann manifold," in *Sixth Int. Symp. on Signal Processing and its Applications*, 2001, vol. 1, pp. 84–87.

[5] A. Srivastava and E. Klassen, "Monte Carlo extrinsic estimators of manifold-valued parameters," *IEEE Trans. on Sig. Proc.*, vol. 50, no. 2, pp. 299–308, Feb. 2002.

[6] B. M. Hochwald, T. L. Marzetta, T. J. Richardson, W. Sweldens, and R. Urbanke, "Systematic design of unitary space-time constellations," *IEEE Trans. Inf. Th.*, vol. 46, no. 6, pp. 1962–1973, Sept. 2000.

[7] L. Zheng and D. N. C. Tse, "Communication on the grassmann manifold: a geometric approach to the noncoherent multiple-antenna channel," *IEEE Trans. Inf. Th.*, vol. 48, no. 2, pp. 359 – 383, Feb. 2002.

[8] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, "Efficient use of side information in mulitiple-antenna data transmission over fading channels," *IEEE Jour. Select. Areas in Commun.*, vol. 16, no. 8, pp. 1423–1436, Oct. 1998.

[9] D. J. Love, R. W. Heath, Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Inf. Th.*, vol. 49, no. 10, pp. 2735 – 2747, Oct. 2003.

[10] K. K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On beamforming with finite rate feedback in multiple antenna systems," *IEEE Trans. Inf. Th.*, vol. 49, no. 10, pp. 2562 – 2579, Oct. 2003.

[11] P. Xia and G. B. Giannakis, "Design and analysis of transmit-beamforming based on limited-rate feedback," in *Proc. Veh. Technol. Conf.*, 2004.

[12] A. Barg and D. Yu. Nogin, "Bounds on packings of spheres in the Grassmann Manifold," *IEEE Trans. Inf. Th.*, vol. 48, no. 9, pp. 2450–2454, Sept. 2002.

[13] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing lines, planes, etc.: Packings in Grassmannian spaces," *Experimental Mathematics*, vol. 5, pp. 139–159, 1996.

[14] B. Mondal and R. W. Heath, Jr., "An upper bound on SNR for limited feedback MIMO beamforming systems," to appear in the *Proc. of IEEE Inf. Th. Workshop*, 2004.

[15] O. Henkel, "Sphere packing bounds in the Grassmann and Stiefel manifolds," submitted to the *IEEE Trans. on Inf. Th.* Available online at http://arxiv.org/abs/math.MG/0308110.

[16] B. M. Hochwald, T. L. Marzetta, and B. Hassibi, "Space-time autocoding," *IEEE Trans. Inf. Th.*, vol. 47, pp. 2761 – 2781, Nov. 2001.

[17] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag Inc., New York, 1999.

[18] B. Mondal, R. W. Heath, Jr., and L. W. Hanlen, "Quantization on the Grassmann manifold with applications to MIMO precoded systems," under preparation.