AN OVERVIEW OF LARGE SYSTEM ANALYSIS FOR MULTI-INPUT MULTI-OUTPUT CHANNELS

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ABSTRACT

Large system analysis has been used extensively in recent years to evaluate the performance of Code Division-Multiple Access (CDMA) and Multi-Input/Multi-Output (MIMO) communications systems. A key feature of this analysis is application of results on eigenvalue distributions and moments of large random matrices. These results enable the efficient computation of large system performance measures, such as spectral efficiency and probability of error, which are far more difficult to compute for finite-size systems. The large system results typically give an accurate prediction of the performance of finite-size systems, and offer important insights into system behavior. We give an overview of large system results for some different communications system models. Our emphasis is on techniques used previously by the authors to evaluate the performance of Multi-Carrier CDMA with the optimal linear receiver.

1. INTRODUCTION

Recent interest in multiuser and multi-antenna wireless systems has motivated numerous performance studies of different variants of multi-input/multi-output (MIMO) channels. These systems include Code-Division Multiple Access (CDMA) (which corresponds to a multi-input/single-output channel) with different receivers, along with different assumptions about the corresponding channel (e.g., ideal or fading), and network (e.g., downlink/uplink cellular or peer-to-peer). Even for relatively simple models, such as a single-user flat fading MIMO channel with the optimal linear receiver, it can be quite difficult to compute performance measures such as probability of error and maximum spectral efficiency averaged over channel realizations. This has motivated *large system* analysis, in which the performance is computed in the limit as certain parameters such as number of transmit and receive antennas, processing gain, and users tend to infinity with fixed ratios.

A key feature of large system analysis is the application of results on eigenvalue distributions for certain large random matrices. Namely, performance measures of interest can often be written in terms of the eigenvalues of a covariance matrix. For the models considered, as the number rows and columns of the covariance matrix increase to infinity in fixed proportion, this set of eigenvalues converges to a deterministic distribution, which can be explicitly computed. The large system performance measures can then be expressed in terms of this distribution (or in some cases, moments of the distribution).

In this paper, we give a brief overview of some recent large system results for CDMA and multi-antenna systems. Comprehensive tutorials on the application of the theory of large random matrices to performance analysis of communications systems are given in [1] and [2]. Those tutorials develop the relevant results on large random matrices from the mathematics literature, and subsequently describe applications to various communication systems models.[1] In contrast, our emphasis in this paper is on some relatively simple techniques, which the authors have used previously to compute the spectral efficiency of Multi-Carrier (MC)-CDMA with the optimal linear receiver. We also indicate how these techniques can be used to analyze the transient behavior of adaptive least squares algorithms for filter estimation.

2. RANDOM MATRIX CHANNEL

We consider a system model in which the received $N \times 1$ vector is given by

$$\mathbf{r} = \mathbf{HSAb} + \mathbf{n} \tag{1}$$

where $\mathbf{b}$ is a $K \times 1$ vector of transmitted symbols, $\mathbf{S}$ is an $N \times K$ matrix, $\mathbf{H}$ is an $N \times N$ channel matrix, $\mathbf{A}$ is a $K \times K$ diagonal amplitude matrix and $\mathbf{n}$ is an $N \times 1$ Gaussian noise vector with covariance $\sigma^2\mathbf{I}$ (scaled identity matrix). This model applies to either a CDMA or multi-antenna system. Namely, for CDMA the columns of $\mathbf{S}$ are

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[1]See [2] for a comprehensive set of references to recent work in this area. Here we refer only to a small subset of directly relevant work.

the user signatures, N is the processing gain, and K is the number of users. For the multi-antenna channel, N and K are the number of receive and transmit antennas, respectively, $\mathbf{S}$ is a precoding matrix, and the channel matrix $\mathbf{H}$ models flat fading across transmit-receive antenna pairs.

In the case of CDMA, the signature matrix $\mathbf{S}$ is typically assumed to be *random*. Namely, for the uplink, the elements of $\mathbf{S}$ are often assumed to be *i.i.d.*, and for the downlink $\mathbf{S}$ is assumed to be a random isotropic matrix so that $\mathbf{S}^\dagger\mathbf{S} = \mathbf{I}$. The channel matrix then models the effect of inter-chip interference. In the case of the flat fading multi-antenna channel, the elements of the channel matrix $\mathbf{H}$ are assumed to be *i.i.d.*.

Finally, we note that for CDMA with inter-chip interference, the corresponding channel matrix is Toeplitz. However, the channel matrix can then be diagonalized through the use of a cyclic prefix, an inverse FFT at the transmitter, and an FFT at the receiver. In what follows, we therefore assume that for CDMA, $\mathbf{H}$ is diagonal.

3. RANDOM SPREADING WITH IDEAL CHANNELS

We start by illustrating large system analysis for a synchronous CDMA system with ideal channels ($\mathbf{H} = \mathbf{I}$) and *i.i.d.* signatures. (Equivalently, a flat fading multi-antenna channel with $\mathbf{S} = \mathbf{I}$.) Let $\mathbf{s}_k$ denote the kth column of $\mathbf{S}$ (signature for user k), $\mathbf{R} = E[\mathbf{r}\mathbf{r}^\dagger] = \mathbf{S}\mathbf{P}\mathbf{S}^\dagger + \sigma^2\mathbf{I}$ denote the received covariance matrix, where $\mathbf{P} = |\mathbf{A}|^2 = \text{diag}[P_1, \cdots, P_K]$, and $\mathbf{R}_k = \mathbf{R} - P_k\mathbf{s}_k\mathbf{s}_k^\dagger$ denote the interference-plus-noise covariance matrix for user k. Then the output Signal-to-Interference Plus Noise Ratio (SINR) for user k with an optimal linear receiver is $\beta_k = P_k\gamma_{-1}$ where $\gamma_{k,n} = \mathbf{s}_k^\dagger\mathbf{R}_k^n\mathbf{s}_k$ can be interpreted as the nth moment of $\mathbf{R}_k$. Ideally, we would like to average this quantity over the signature matrix $\mathbf{S}$ to obtain a performance measure that depends only on N, K, and the distribution of P_k. This appears to be quite difficult; however, we observe that

$$\begin{aligned} 1 &= \frac{1}{N}\text{trace }(\mathbf{R}^{-1}\mathbf{R}) \\ &= \frac{1}{N}\text{trace }\left[\mathbf{R}^{-1}(\sigma^2\mathbf{I} + \mathbf{S}\mathbf{P}\mathbf{S}^\dagger)\right] \\ &= \sigma^2\frac{1}{N}\text{trace }\mathbf{R}^{-1} + \sum_{k=2}^{K}\frac{1}{N}\frac{P_k\gamma_{k,-1}}{1 + P_k\gamma_{k,-1}} \end{aligned} \qquad (2)$$

where we have used the matrix inversion lemma.

Now suppose we let the pair $(K, N) \to \infty$ with fixed $\alpha = K/N$. It turns out that $\gamma_{k,n}$ then converges in probability to a *deterministic* limit γ_n^∞ [3–5]. Note that γ_n^∞ is independent of k, reflecting the fact that for a large system the distribution of interfering signatures is the same for each user. Furthermore, we also observe that because $\gamma_n^\infty = \lim_{(K,N)\to\infty} E_\mathbf{S}[\gamma_{k,n}]$, and $\mathbf{s}_k$ is independent of $\mathbf{R}_k$, we must have $\gamma_n^\infty = \lim_{(K,N)\to\infty} \frac{1}{N}\text{trace }\mathbf{R}^n$. Hence from (2) we have

$$1 = \gamma_{-1}^\infty\left(\sigma^2 + \alpha\int\frac{P}{1 + P\gamma_{-1}^\infty}dF(P)\right) \qquad (3)$$

where $F(\cdot)$ is the distribution of the powers over users.

This fixed point equation was first presented in [5], and was derived by applying results from [3]. The preceding derivation is presented in [6].[2]

We remark that replacing σ^2 in the preceding derivation by the complex variable z gives the fixed-point equation for the Stieltjes transform of the asymptotic ($(K, N) \to \infty$) eigenvalue distribution of $\mathbf{R}$ (ignoring associated convergence issues). The Stieltjes transform is essentially equivalent to the z-transform of the sequence of moments of the asymptotic eigenvalue distribution, and can be inverted to obtain the distribution.

The asymptotic eigenvalue distribution (AED) of $\mathbf{R}$ can be used to evaluate large system spectral efficiency for the optimal (maximum likelihood) receiver [4]. Application of large system analysis to other multiuser receivers (e.g., decorrelating and decision feedback) for the channel model (1) are described in [1, 2].

4. NON-IDEAL CHANNELS

We now consider (1) with general $\mathbf{H}$, in which case the asymptotic output SINR for user k with an optimal linear receiver is $\beta_k = P_k\rho_{k,-1}$, where $\rho_{k,n} = \mathbf{s}_k^\dagger\mathbf{H}\mathbf{R}_k^n\mathbf{H}^\dagger\mathbf{s}_k$, $\mathbf{R} = \mathbf{H}\mathbf{S}\mathbf{P}\mathbf{S}^\dagger\mathbf{H}^\dagger + \sigma^2\mathbf{I}$, and $\mathbf{R}_k = \mathbf{R} - P_k\mathbf{H}\mathbf{s}_k\mathbf{s}_k^\dagger\mathbf{H}^\dagger$. We are therefore interested in computing $\lim_{(N,K)\to\infty}\rho_{k,-1} = \rho_{-1}^\infty$. It can be shown that [7, Proposition 4]

$$\rho_n^\infty = \lim_{(N,K)\to\infty}\begin{cases}\frac{1}{N}\text{tr}[\mathbf{H}^\dagger\mathbf{R}^n\mathbf{H}] & , iid \\ \frac{1}{N-K}\text{tr}[\mathbf{\Pi}\mathbf{H}^\dagger\mathbf{R}^n\mathbf{H}] & , \text{isometric.}\end{cases} \qquad (4)$$

and $\mathbf{\Pi} = \mathbf{I} - \mathbf{S}\mathbf{S}^\dagger$ where *iid* and isometric refer to the signature matrix $\mathbf{S}$.

It is possible to use the same trick as in (2) to obtain

$$1 = \sigma_n^2\gamma_{-1}^\infty + \alpha\rho_{-1}^\infty\int\frac{P}{1 + P\rho_{-1}^\infty}dF(P) \qquad (5)$$

where again $\gamma_{-1}^\infty = \lim_{(K,N)\to\infty}\frac{1}{N}\text{tr}[\mathbf{R}^{-1}]$. In this case, we can calculate $\gamma_{-1}^\infty = \mathsf{E}\left[\frac{1}{\lambda + \sigma_n^2}\right]$, where λ is a scalar random variable with distribution given by the AED of $\mathbf{H}\mathbf{S}\mathbf{P}\mathbf{S}^\dagger\mathbf{H}^\dagger$. That AED can be computed by using the S-transform, which arises in the theory of free probability (for both *iid* and isometric signatures). Namely, for asymptotically free matrices $\mathbf{A}$ and $\mathbf{B}$, the S-transform can be used to compute the

[2] Strictly speaking, this derivation is not rigorous since it is based on the assumption that $\gamma_{k,-1}$ converges uniformly over k to a deterministic limit.

AED of the product $\mathbf{AB}$ in terms of the AED's of $\mathbf{A}$ and $\mathbf{B}$ (see [2]). In our case, because $\mathbf{A} = \mathbf{H}$ is diagonal and $\mathbf{B} = \mathbf{S}$ has *iid* elements, they are asymptotically free provided that the diagonal components of $\mathbf{H}$ are bounded [2].

Alternative derivations of the maximum SINR for the model (1) with a linear receiver and *iid* signatures are given in [8], [9], and [10]. In particular, we can rewrite $\rho_{-1}^{\infty} = \lim_{(N,K)\to\infty} \frac{1}{N}\mathrm{tr}[(\mathbf{SPS}^{\dagger} + \sigma_n^2(\mathbf{H}^{\dagger}\mathbf{H})^{-1})^{-1}]$, and note that the AED of the matrix within the trace is given in [3]. The previous approach has the advantage of applying to both *iid* and isometric signatures, and to non-square channel matrices $\mathbf{H}$.

5. MULTI-SIGNATURE CDMA

One way to increase the data rate in CDMA is to encode symbols over multiple signatures. In that case the model (1) becomes

$$\mathbf{r} = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{S}_k \mathbf{b}_k + \mathbf{n} \tag{6}$$

where k is the user index, and $\mathbf{S}_k$ is an $N \times J_k$ random i.i.d. spreading matrix, where J_k is the number of signatures assigned to user k. The total number of signatures is $J = \sum_{k=1}^{K} J_k$. This model also applies to a multi-access MIMO channel where $\mathbf{S}_k$ is the precoding matrix for user k.

Here we are interested in evaluating how performance depends on the number of signatures assigned to the different users. We therefore *fix* the number of users, and let J_k, $k = 1, \cdots, K$, and N all tend to infinity with fixed ratios $\alpha_k = J_k/N$. We can then compute a spectral efficiency *region*. That is, for two users we can compute the maximum achievable rate for user 2, given an achievable rate for user 1.

The output SINR for signature j of user k with an optimal linear receiver is $\beta_{j,k} = \rho_{j,k,-1}$, where $\rho_{j,k,n} = \mathbf{s}_{j,k}^{\dagger}\mathbf{H}_k\mathbf{R}_{j,k}^{n}\mathbf{H}_k^{\dagger}\mathbf{s}_{j,k}$, $\mathbf{R} = \sum_{k=1}^{K} \mathbf{H}_k\mathbf{S}_k\mathbf{S}_k^{\dagger}\mathbf{H}_k^{\dagger} + \sigma^2\mathbf{I}$, and $\mathbf{R}_{j,k} = \mathbf{R} - \mathbf{H}_k\mathbf{s}_{j,k}\mathbf{s}_{j,k}^{\dagger}\mathbf{H}_k^{\dagger}$. Moreover, it can be shown that $\lim \rho_{j,k,n} = \rho_{k,n}^{\infty}$, which is independent of the signature j, and

$$\rho_{k,n}^{\infty} = \lim \frac{1}{N}\mathrm{tr}[\mathbf{H}_k^{\dagger}\mathbf{R}^{n}\mathbf{H}_k]$$

where $\lim$ denotes the limit $(N, J_u) \to \infty$ with $J_u/N \to \alpha_u$ for all $u = 1, \ldots, K$.

The large system SINR $\rho_{k,n}^{-1}$ can again be computed in terms of $\gamma_{-1}^{\infty} = \lim \gamma_{-1}$, where $\gamma_{-1} = \frac{1}{N}\mathrm{tr}[\mathbf{R}^{-1}]$. This computation is not a straightforward extension of the previous results due to the more complicated structure of $\mathbf{R}$. One approach is to use the following *incremental signature* technique. Namely, we consider the change in γ_{-1} when a single signature $\mathbf{s}$ is added to user u. The load α_u is then replaced by $\alpha_u + 1/N$, i.e., $\triangle\alpha_u = 1/N$. Applying the matrix inversion lemma, the value of $\gamma_{-1}(\alpha_u)$ with the additional signature can be evaluated as

$$\gamma_{-1}(\alpha_u + \triangle\alpha_u) = \frac{1}{N}\mathrm{tr}[\left(\mathbf{R} + \mathbf{H}_u\mathbf{s}\mathbf{s}^{\dagger}\mathbf{H}_u^{\dagger}\right)^{-1}]$$

so that

$$\gamma_{-1}(\alpha_u + \triangle\alpha_u) - \gamma_{-1}(\alpha_u) = -\frac{\mathbf{s}^{\dagger}\mathbf{H}_u^{\dagger}\mathbf{R}^{-2}\mathbf{H}_u\mathbf{s}}{1 + \mathbf{s}^{\dagger}\mathbf{H}_u^{\dagger}\mathbf{R}^{-1}\mathbf{H}_u\mathbf{s}} \triangle\alpha_u$$

which in the large system limit previously defined gives

$$\frac{\partial}{\partial\alpha_u}\gamma_{-1}^{\infty} = -\frac{\rho_{u,2}^{\infty}}{1 + \rho_{u,1}^{\infty}} \tag{7}$$

Additional relations can be derived by expanding the trace of $\mathbf{RR}^{-1}$ and also $\mathbf{RR}^{-2}$ as in Section 3, and combining those with (7) gives

$$1 = \sigma_n^2\gamma_{-1}^{\infty} + \sum_{k=1}^{K}\alpha_k\frac{\rho_{k,-1}^{\infty}}{(1 + \rho_{k,-1}^{\infty})} \tag{8}$$

$$\gamma_{-1}^{\infty} = \sigma_n^2\gamma_{-2}^{\infty} - \sum_{k=1}^{K}\alpha_k\frac{1}{(1 + \rho_{k,-1}^{\infty})}\frac{\partial\gamma_{-1}^{\infty}}{\partial\alpha_k} \tag{9}$$

For $K = 2$ (two users) the SINRs for each user can therefore be computed if γ_{-1}^{∞}, γ_{-2}^{∞} and $\frac{\partial\gamma_{-1}^{\infty}}{\partial\alpha_k}$ are known. We have not been able to compute those quantities exactly, but note that they can be approximated easily if $\mathbf{H}_k$ is approximated by $\mathbf{U}_k\mathbf{H}_k$, where $\{\mathbf{U}_k\}_{k=1,\ldots,K}$ are unitarily invariant random independent matrices. In that case, the matrices being summed in $\mathbf{R}$ are asymptotically free, and it is possible to compute the AED of $\mathbf{R}$ (and therefore γ_{-1}) through the use of the R-transform. Namely, given two matrices $\mathbf{A}$ and $\mathbf{B}$, which are asymptotically free, the R transform can be used to compute the AED of the sum $\mathbf{A} + \mathbf{B}$ in terms of the AED's of $\mathbf{A}$ and $\mathbf{B}$ [2]. Results in [11] show that the preceding "free approximation" is quite accurate for a wide range of system parameters (loads, SNRs).

We remark that a different set of equations, which also relate the SINR with γ_{-1}^{∞}, can be obtained using the S-transform [11].

These analytical results can be used to compare the performance of MC-CDMA with different sets of loads ($\{\alpha_k\}$) across users, and also to compare MC-CDMA with Orthogonal Frequency Division Multiple Access (OFDMA). OFDMA has higher spectral efficiency than MC-CDMA, since the users do not interfere, but requires more coordination since users must be assigned to specific tones.

6. ADAPTIVE LEAST SQUARES (LS) ESTIMATION

Large system analysis has also been used to study the transient behavior of adaptive LS receivers for the signal models considered in Sections 3 and 4 [6]. The LS filter replaces the covariance matrix $\mathbf{R}$ with the *sample* covariance matrix $\hat{\mathbf{R}}$. The transient behavior of the output SINR, i.e., SINR as a function of training symbols (assuming a training sequence is available), or observations (assuming the desired user's signature and channel are known without training) can be computed in the large system limit by letting $(K, N, t) \to \infty$ with fixed ratios K/N and t/N where t is the number of training symbols or observations. Evaluation of output SINR then depends on computing the AED of the *sample* covariance matrix in this limit. This can be accomplished using the techniques previously described. Furthermore, it possible to include important algorithmic features in the analysis, such as diagonal loading (to prevent ill-conditioning of the sample covariance matrix), and data windowing (e.g., exponential weighting to discount past observations).

7. OTHER MODELS AND FUTURE WORK

Large system analysis, and the application of results from large random matrix theory, has been used to evaluate the performance of many other types receivers and communications systems. For example, reduced-rank receivers, in which the input signal is projected onto a lower dimensional subspace, can be analyzed using this framework [2]. The large system performance of iterative receivers is evaluated in [12, 13]. Large system analysis has also been used to evaluate the performance of multi-antenna systems with correlated fading [2].

Evaluation of the *minimum* bit error rate for CDMA, corresponding to the optimal *maximum a posteriori* detector, has been evaluated in the large system limit through an application of the replica method in statistical physics [14]. This method is quite different from the methods described here, and may be useful in other related contexts. Other applications of large system analysis to signal processing and communications techniques are still emerging.

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