On geolocation in ill-conditioned BS-MS layouts

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***Abstract*— The achievable accuracy for estimating the position of a mobile station (MS) in terms of the Cramer-Rao Lower bound (CRLB) has been well studied and accepted. For certain class of layouts of an MS and base stations (BSs), termed as ill-conditioned layouts here, the CRLB suggests that the variance of estimation errors is infinitely large. In this paper, we investigate how to overcome this problem in both analytical and algorithmic aspects. The ill-conditioned layouts are classified into two categories: layouts with an insufficient number of BSs and those satisfying the condition of collinearity or coplanarity. We investigate estimation schemes and error analysis for each category. Accordingly, a geolocation method with adaptive dimensions is devised. Simulation results confirm effectiveness of our method.**

I. Introduction

The positioning accuracy[1] in terms of the Cramer-Rao Lower bound (CRLB) has been well accepted in the literature [1], [2]. For certain base station (BS) and mobile station (MS) layouts, termed as *ill-conditioned* layouts herein, the CRLB suggests that the variance of estimation errors becomes infinitely large. A least square (LS) based solution using ridge regression is proposed in [5], where the basic idea is to introduce a small bias for achieving a significant variance reduction.

In this paper, we pursue this problem in an intuitive manner. First, ill-conditioned circumstances are classified into two categories: one with an insufficient number of BSs and the other satisfying the condition of *collinearity* or *coplanarity* which is to be defined in Section II. Then, by using the time-of-arrival (TOA) based method, we investigate estimation schemes and corresponding mean square errors (MSE) for each category. It is shown that finite variance can be obtained, instead of infinitely large variance suggested by the CRLB. Next, a geolocation method with adaptive dimensions is proposed to incorporate the analytical results. Simulation examples are examined lastly.

The rest of the paper is organized as follows. Section II introduces preliminaries of this topic. In Section III, we present main analytical results. In Section IV, a geolocation scheme is devised to improve the positioning accuracy in ill-conditioned BS-MS layouts. Simulation examples are discussed in Section V. A brief conclusion is made in last section.

[1]The geometric dilution of precision (GDOP) is also a well-discussed accuracy measure [3]. It can be derived as the CRLB with Gaussian measurement errors [4].

II. Preliminaries

Consider TOA positioning, where BSs receive a radio signal transmitted from an MS. The MS position is to be estimated based on propagation delays measured by the BSs. Denote the MS position by vector $\boldsymbol{p}$, where $\boldsymbol{p} = (x,\ y)^T$ or $(x,\ y,\ z)^T$ depends on problem formulation. The superscript "T" stands for transpose. Let $\mathcal{B} = \{1, 2, \cdots, B\}$ be the set of indices of the BSs, whose locations $\{\boldsymbol{p}_b,\ b \in \mathcal{B}\}$ are known. A delay (or TOA) estimate obtained at the b-th base station (BS$_b$) can be written as

$$\hat{\tau}_b = \tau_b + \varepsilon_b, \quad \text{for } b \in \mathcal{B}, \tag{1}$$

where $\tau_b = \|\boldsymbol{p}_b - \boldsymbol{p}\|$ is the distance between the MS and BS$_b$, and the estimation error ε_b can be approximated by a Gaussian random variable $\mathcal{N}(0, \sigma_b^2)$. Note that the above equation is normalized with respect to the speed of light $c = 3 \times 10^8$m/s, thus is expressed in the unit of length. In practice, the delay estimation can be implemented, e.g., by Two-way Time Transfer technique or Double Token Exchange TOA technique [6].

The Cramer-Rao Lower Bound (CRLB) sets a lower limit for variance (or a covariance matrix) of any unbiased estimate of an unknown parameter (or unknown parameters) [7]. The positioning accuracy in terms of the CRLB has been well discussed in the literature [1], [2], [8]. Denote an estimate of $\boldsymbol{p}$ by $\hat{\boldsymbol{p}}$. The CRLB matrix is defined as the inverse of the *Fisher information matrix* (FIM)

$$E_{\boldsymbol{p}}\left[(\hat{\boldsymbol{p}} - \boldsymbol{p})(\hat{\boldsymbol{p}} - \boldsymbol{p})^T\right] \geq \mathbf{J}_{\boldsymbol{p}}^{-1}, \tag{2}$$

where "$\mathbf{A} \geq \mathbf{B}$" should be interpreted as non-negative definiteness of matrix $(\mathbf{A} - \mathbf{B})$, and $E_{\boldsymbol{p}}[\cdot]$ stands for the expectation conditioned on $\boldsymbol{p}$. It can be shown [8] that

$$\mathbf{J}_{\boldsymbol{p}} = \mathbf{H} \cdot \mathbf{\Lambda} \cdot \mathbf{H}^T, \tag{3}$$

where

$$\mathbf{\Lambda} = \text{diag}\left(\sigma_1^{-2},\ \sigma_2^{-2},\ \cdots, \sigma_B^{-2}\right), \tag{4}$$

and the matrix $\mathbf{H}$ in 2-D and 3-D settings are

$$\mathbf{H}_{2D} = \begin{pmatrix} \cos\phi_1 & \cos\phi_2 & \cdots & \cos\phi_B \\ \sin\phi_1 & \sin\phi_2 & \cdots & \sin\phi_B \end{pmatrix}, \tag{5}$$

and

$$\mathbf{H}_{3D} = \begin{pmatrix} \cos\theta_1\cos\phi_1 & \cos\theta_2\cos\phi_2 & \cdots & \cos\theta_B\cos\phi_B \\ \cos\theta_1\sin\phi_1 & \cos\theta_2\sin\phi_2 & \cdots & \cos\theta_B\sin\phi_B \\ \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_B \end{pmatrix}, \quad (6)$$

respectively. The angle ϕ_b and θ_b are determined by the positions of the MS and BS$_b$ as $\phi_b = \tan^{-1}\frac{y-y_b}{x-x_b}$, and

$$\theta_b = \sin^{-1}\frac{z-z_b}{\sqrt{(x-x_b)^2+(y-y_b)^2+(z-z_b)^2}}.$$

Note that $\phi_b \in [0, 2\pi)$ and $\theta_b \in [-\pi/2, \pi/2]$. The minimum mean square error (MMSE) can be evaluated as the trace of the CRLB matrix

$$\mathcal{P} \overset{\text{def}}{=} \left[E\|\boldsymbol{p}-\widehat{\boldsymbol{p}}\|^2\right]_{\min} = \text{trace}\left(\mathbf{J}_{\boldsymbol{p}}^{-1}\right). \quad (7)$$

It is not difficult to show that $\mathcal{P}$ is infinitely large due to the rank deficiency of the FIM in the following two categories of MS-BS layouts, independent of the type of coordinate chosen:

1) A layout with an insufficient number of BSs. That is, there is only one BS in two-dimension (2-D) geolocation, and one or two BSs in a 3-D setting.
2) A layout with the condition of collinearity in a 2-D or 3-D formulation or coplanarity in a 3-D formulation. The collinearity (or coplanarity) means that all BSs and an MS of interest are located on a straight line (or on a plane).

III. Analysis for ill-conditioned MS-BS layouts

In this section, concrete estimation schemes are developed and corresponding MSEs are derived for each category of the ill-conditioned layouts specified in the previous section.

A. Category 1: layouts with an insufficient number of BSs

A.1. One BS in a 2-D formulation

When only one delay estimate, say $\hat{\tau}_1$, is available, an MS position estimate can be anywhere on the circle with radius $\hat{\tau}_1$ and center (x_1, y_1). Specifically, a polar coordinate with origin (x_1, y_1) is adopted for simplicity. Denote the position estimate by $(\hat{\rho}, \hat{\psi})$. The estimation scheme is that $\hat{\rho} = \hat{\tau}_1$ and angle $\hat{\psi}$ is selected according to the uniform distribution within $[0, 2\pi)$. Note that $\hat{\rho}$ is a Gaussian random variable with $\mathcal{N}(\tau_1, \sigma_1^2)$. Hence, the MSE can be derived as

$$\begin{aligned} & E\|\boldsymbol{p}-\widehat{\boldsymbol{p}}\|^2 \\ = \; & \frac{1}{(2\pi)^{3/2}\sigma_1}\int_0^{+\infty}\int_0^{2\pi}\Big[(\hat{\rho}\cos\hat{\psi}-x)^2 \\ & +(\hat{\rho}\sin\hat{\psi}-y)^2\Big]\cdot\exp\left\{-\frac{1}{2\sigma_1^2}(\hat{\rho}-\tau_1)^2\right\}\,d\hat{\psi}\,d\hat{\rho} \\ = \; & \frac{1}{\sqrt{2\pi}\sigma_1}\int_0^{\infty}\left[\hat{\rho}^2+\tau_1^2\right]\exp\left\{-\frac{1}{2\sigma_1^2}(\hat{\rho}-\tau_1)^2\right\}d\hat{\rho} \\ \approx \; & \frac{1}{\sqrt{2\pi}\sigma_1}\int_{-\infty}^{\infty}\left[\hat{\rho}^2+\tau_1^2\right]\exp\left\{-\frac{1}{2\sigma_1^2}(\hat{\rho}-\tau_1)^2\right\}d\hat{\rho} \\ & (\text{assuming } \tau_1 >> \sigma_1) \\ = \; & 2\tau_1^2+\sigma_1^2 < \infty, \end{aligned} \quad (8)$$

which is a finite value. In some situations, the angle $\hat{\psi}$ can be confined to certain region denoted by $[-\alpha,\ \alpha]$ with $\alpha \in [0, \pi)$. Then the MSE can be improved as

$$\begin{aligned} & E\|\boldsymbol{p}-\widehat{\boldsymbol{p}}\|^2 \\ = \; & \frac{1}{\sqrt{2\pi}\sigma_1}\int_0^{+\infty}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\Big[(\hat{\rho}\cos\hat{\psi}-x)^2 \\ & +(\hat{\rho}\sin\hat{\psi}-y)^2\Big]\cdot\exp\left\{-\frac{1}{2\sigma_1^2}(\hat{\rho}-\tau_1)^2\right\}\,d\hat{\psi}\,d\hat{\rho} \\ \approx \; & 2\tau_1^2+\sigma_1^2-2x\tau_1\sin\alpha/\alpha. \end{aligned} \quad (9)$$

Note the hybrid TOA/AOA(angle-of-arrival) scheme with one BS can be seen as a special case of the one-BS geolocation scheme discussed here, except that ψ usually follows a Gaussian distribution.

A.2. One BS in a 3-D formulation

Similarly, we adopt a spherical coordinate with the origin at BS$_1$'s location for one-BS 3-D geolocation. Let the true MS position be $(\rho,\ \theta,\ \psi)$. The position estimate is on the sphere of radius $\hat{\rho} = \hat{\tau}_1$ with $\hat{\theta}$ and $\hat{\psi}$ uniformly distributed within $[0,\ \pi]$ and $[0,\ 2\pi)$, respectively. The MSE is derived as

$$\begin{aligned} & E||\hat{\boldsymbol{p}}-\boldsymbol{p}||^2 \\ = \; & \frac{1}{2^{3/2}\pi^{5/2}\sigma_1}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\Big[(\hat{\rho}\cos\hat{\theta}-\rho\cos\theta)^2 \\ & +(\hat{\rho}\sin\hat{\theta}\cos\hat{\psi}-\rho\sin\theta\cos\psi)^2+(\hat{\rho}\sin\hat{\theta}\sin\hat{\psi} \\ & -\rho\sin\theta\sin\psi)^2\Big]\exp\left\{-\frac{1}{2\sigma_1^2}(\hat{\rho}-\tau_1)^2\right\}\,d\hat{\psi}\,d\hat{\theta}\,d\hat{\rho} \\ \approx \; & 2\tau_1^2+\sigma_1^2. \end{aligned} \quad (10)$$

When additional information is available to limit $\hat{\psi} \in [-\mu,\ \mu]$ and $\hat{\theta} \in [\nu,\ \pi-\nu]$ with $\mu \in [0, \pi)$ and $\nu \in [0, \pi/2)$, the MSE is modified as

$$\begin{aligned} & E||\hat{\boldsymbol{p}}-\boldsymbol{p}||^2 \\ = \; & \frac{1}{\sqrt{2\pi}\sigma_1}\int_0^{+\infty}\frac{1}{\pi-2\nu}\int_{\nu}^{\pi-\nu}\frac{1}{2\mu}\int_{-\mu}^{\mu}\Big[(\hat{\rho}\cos\hat{\theta}-\rho\cos\theta)^2 \\ & +(\hat{\rho}\sin\hat{\theta}\cos\hat{\psi}-\rho\sin\theta\cos\psi)^2+(\hat{\rho}\sin\hat{\theta}\sin\hat{\psi} \\ & -\rho\sin\theta\sin\psi)^2\Big]\exp\left\{-\frac{1}{2\sigma_1^2}(\hat{\rho}-\tau_1)^2\right\}\,d\hat{\psi}\,d\hat{\theta}\,d\hat{\rho} \\ \approx \; & 2\tau_1^2+\sigma_1^2-\frac{4\cos\nu\sin\mu}{\mu(\pi-2\nu)}x\tau_1, \end{aligned} \quad (11)$$

where

$$x = \rho\sin\theta\cos\psi.$$

A.3. Two BSs in a 3-D formulation

Suppose two delay estimates, $\hat{\tau}_1$ and $\hat{\tau}_2$, are available. We first review a 2-D solution. Without loss of generality, let $\boldsymbol{p}_1 = (x_1,\ 0)^T$ and $\boldsymbol{p}_2 = (x_2,\ 0)^T$, and an MS position be $\boldsymbol{p} = (x,\ y)$. A position estimate $\hat{\boldsymbol{p}} = (\hat{x},\ \hat{y})^T$ conforms to the joint Gaussian p.d.f conditioned on $\boldsymbol{p}$:

$$f(\hat{\boldsymbol{p}}|\boldsymbol{p}) = \frac{|\mathbf{J}_{\boldsymbol{p}}|^{1/2}}{2\pi}\exp\left\{-\frac{1}{2}(\hat{\boldsymbol{p}}-\boldsymbol{p})^T\mathbf{J}_{\boldsymbol{p}}(\hat{\boldsymbol{p}}-\boldsymbol{p})\right\}, \quad (12)$$

where $\mathbf{J}_{\boldsymbol{p}}$ is the FIM given in Eq. (3) with $\mathbf{H} = \mathbf{H}_{2D}$.

Re-formulate the problem in a 3-D setting using a cylindrical coordinate. The MS position is then given by (ρ, ψ, z) with $z = x$ and $\rho = y$. An estimate scheme is devised as $\hat{\rho} = \hat{y}$, $\hat{z} = \hat{x}$ and $\hat{\psi}$ is selected according to the uniform distribution within $[0, 2\pi)$. It can be shown that the corresponding MSE is

$$\begin{aligned} & E||\hat{\boldsymbol{p}} - \boldsymbol{p}||^2 \\ = & \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{2\pi} \Big[(\hat{\rho}\cos\hat{\psi} - \rho\cos\psi)^2 + (\hat{\rho}\sin\hat{\psi} \\ & -\rho\sin\psi)^2 + (\hat{z} - z)^2 \Big] \cdot f(\hat{\boldsymbol{p}}|\boldsymbol{p}) \, d\hat{\psi} \, d\hat{\rho} \, d\hat{z} \\ = & \sigma_x^2 + \sigma_y^2 + 2y^2, \end{aligned} \tag{13}$$

where

$$\sigma_x^2 = \left[\mathbf{J}_{\boldsymbol{p}}^{-1}\right]_{11}, \quad \sigma_y^2 = \left[\mathbf{J}_{\boldsymbol{p}}^{-1}\right]_{22}, \tag{14}$$

and $[\mathbf{A}]_{nn}$ means the n-th diagonal term of matrix $\mathbf{A}$.

B. Category 2: layouts satisfying the condition of collinearity or coplanarity

B.1. A collinear scenario

When all BSs and an MS are aligned, the geolocation is essentially a one-dimension problem. Accordingly, we modify the delay estimates in Eq. (1) as

$$\hat{\tau}_b = \pm(x - x_b) + \varepsilon_b, \quad \text{for } b \in \mathcal{B}, \tag{15}$$

where x_b and x are the positions of BS_b and the MS, respectively. It is assumed that there is sufficient information to determine the sign in front of $(x - x_b)$ in the above equation. The maximum likelihood estimate of the MS position can be shown as a weighted sum

$$\hat{x} = \sum_{b=1}^{B} w_b \cdot (x_b \pm \hat{\tau}_b), \tag{16}$$

where

$$w_b = \frac{1/\sigma_b^2}{\sum\limits_{i=1}^{B} 1/\sigma_i^2}. \tag{17}$$

The MSE is

$$\text{var}(\hat{x}) = \sum_{b=1}^{B} w_b^2 \cdot \text{var}(\hat{\tau}_b) = \frac{1}{\sum_{b=1}^{B} 1/\sigma_b^2}. \tag{18}$$

B.2. A coplanar scenario

First identify the plane where the BSs and the MS locate, and then apply a conventional 2-D scheme. The MSE is given in terms of the CRLB as

$$\begin{aligned} & E||\hat{\boldsymbol{p}} - \boldsymbol{p}||^2 = \text{trace}\left(\mathbf{J}_{\boldsymbol{p}}^{-1}\right), \\ = & \frac{c^2}{8\pi^2\beta^2} \cdot \frac{\sum\limits_{b \in \mathcal{B}} R_b}{\sum\sum\limits_{b_1, b_2 \in \mathcal{B}} R_{b_1} R_{b_2} \sin^2(\phi_{b_1} - \phi_{b_2})}, \end{aligned} \tag{19}$$

where $\mathbf{J}_{\boldsymbol{p}}$ is given in Eq. (3) with $\mathbf{H} = \mathbf{H}_{2D}$.

C. Discussions

There is an essential difference between the above two categories of geolocation. In the first category, the geolocation itself is a "full dimension" problem, yet observations (i.e., delay estimates) are only sufficient to locate the MS in some dimensions, while leaving estimation in the remaining dimension(s) completely or partly unconstrained. For example, in the case A.1, the radius $\hat{\rho}$ can be determined as $\hat{\tau}_1$, but the angle ψ can be any value within $[0, 2\pi)$ or $[-\alpha, \ \alpha]$. In contrast, geolocation in the second category is a reduced-dimension problem. Reformulation in appropriate dimensions is the solution.

IV. A Geolocation Scheme with adaptive dimensions

By incorporating the analytical results obtained in the previous section, we propose a geolocation scheme with adaptive dimensions for ill-conditioned MS-BS layouts. Specifically, the scheme includes five steps:

1) Determine if there are a sufficient number of BSs. Typically, two and three BSs are the least requirements for 2-D and 3-D positioning, respectively.
2) If there are not enough BSs, which belongs to Category 1, we adopt the estimation schemes specified in Section III-A. The corresponding MSE is given in Eq. (8), (11) or (13), which can be used to examine whether the current estimate satisfies certain threshold or not. Such a threshold depends on the type of geolocation service, say 100m for E-911 service.
3) If there are a sufficient number of BSs, examine whether the BSs are collinear or coplanar.
4) If the BSs are collinear or coplanar, we further check whether the MS is in the reduced-dimension space (i.e., the line or the plane determined by the BSs) or its neighborhood according to one of the following three types of information:
   - Prior information, such as position and speed estimates in an earlier instant in a position tracking procedure.
   - Unfeasible converging points when a conventional "full-dimension" scheme is applied.
   - Unreliable computation when a gradient-based optimization technique is adopted, which is due to inverse of some rank-deficient matrix.
5) If the BSs and MS are collinear or coplanar, reformulate the problem with appropriate dimensions. Otherwise employ a conventional scheme.

Besides the improved positioning accuracy, the major advantages of the proposed approach include reduced computational complexity and predictable positioning errors.

V. Simulation Results

In this section, we investigate simulation results for two dimensional geolocation. The square root of the MSE (RMSE) is adopted as the measure of positioning errors. One thousand simulation runs are executed to evaluate the RMSE.

A. Geolocation with one BS

Assume there is only one BS (BS_1) available. Let the positions of BS_1 and an MS be $(0, 0)$ and $(200, 0)$, respectively. The scheme given in Section III-A is used. The standard deviation of the delay estimate $\hat{\tau}_1$, σ_1, is set to be 10m. Figure 1 illustrates the performance curve of the RMSE vs. α together with the curve determined by the analytical result of Eq. (9) . The performance curve is well predicted by the analytical result. It is seen that the positioning error RMSE can be small, say less than 100m, in a one-BS layout, if α is below certain value, e.g., $60°$ in this case.

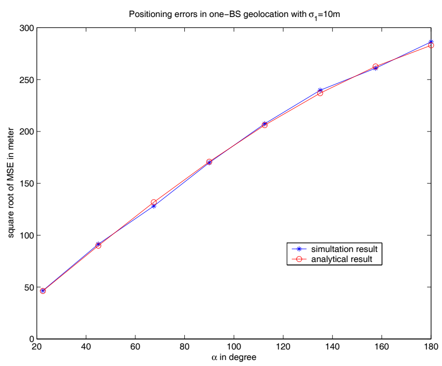

Fig. 1. The performance curve of the RMSE (in meter) vs. angle α (in degree) with $\sigma_1 = 10$m, denoted by the line with "$*$", compared with the curve corresponding to the analytical curve with "$\circ$".

B. Geolocation with quasi-collinearity of the BSs and an MS

Assume four BSs are lined up with location coordinates $(-3000, 0)$, $(-1000, 0)$, $(1000, 0)$ and $(3000, 0)$. An MS is at $(600, 8)$, which is close to the line $y = 0$. Let the standard deviation of a delay estimate at the distance of 2000m from the MS be σ_0. Deviations of delay estimates at the BSs are evaluated with path loss factor 2 accordingly.

Two schemes are adopted for position estimation. One is the conventional 2-D maximum likelihood (ML) method. The other is the 1-D scheme described in Section III-A.3. The initial estimate $(600, 18)$ is used in searching steps for the 2-D scheme. Figure 2 shows the RMSE vs. σ_0 using the two schemes compared with the curve determined by the analytical result of Eq. (18). It is seen that the 1-D scheme can achieve a better position accuracy than the conventional 2-D scheme. In addition, since the 1-D estimation has an algebraic solution, it requires a much less computational load than the 2-D scheme that involves iterative searching steps.

VI. Conclusions

In this paper, we investigate geolocation schemes and accuracy in ill-conditioned MS-BS layouts. Instead of infinitely large error variance suggested by the CRLB, we show that finite (and sometimes rather small) error variance can be obtained when appropriate schemes are applied. A geolocation method with adaptive dimensions is proposed to improve the positioning accuracy in the ill-conditioned layouts. Its effectiveness is confirmed by simulation results.

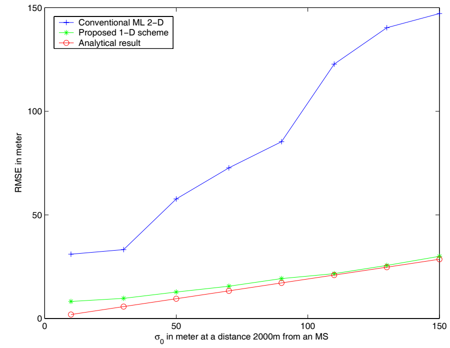

Fig. 2. The performance curves of the RMSE (in meter) vs. σ_0 (in meter) using the conventional ML 2-D scheme (the top curve) and the proposed 1-D scheme (the curve in the middle), compared with the curve corresponding to the analytical result (the bottom curve).

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