ACCURACY EVALUATION OF FIXED-POINT APA ALGORITHM

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ABSTRACT

The implementation of adaptive filters with fixed-point arithmetic requires to evaluate the computation quality. The accuracy can be determined by calculating the global quantization noise power in the system output. In this paper, a new model for evaluating analytically the global noise power in the APA algorithm is developed. The model is presented and applied to the NLMS-OCF. The accuracy of our model is analyzed by experimentations.

1. INTRODUCTION

The aim of adaptive filters is to estimate a sequence of scalars from an observation sequence filtered by a system in which coefficients vary. These coefficients converge towards the optimum coefficients which minimize the mean square error (MSE) between the filtered observation signal and the desired sequence. This type of filters is used in different fields such as noise cancellation, equalization, linear prediction and channel estimation. The different algorithms for adaptive filtering are mainly classified in two types : Recursive Least Square (RLS) and Least Mean Square (LMS). Nevertheless, the LMS algorithm is the most common used in embedded real-time applications because its implementation is more simple than the RLS algorithm. However, the Affine Projection Algorithms (APA) have been developed very recently [3] to have a faster convergence compared to the LMS and to reduce complexity compared to RLS. The convergence behavior of this algorithm has been studied in [4] and [5] but no study is available of its fixed-point implementation. For embedded systems, the use of fixed-point arithmetic is required because it is less expensive in terms of cost and power consumption than the floating-point arithmetic. But, the fixed-point processing introduces an error called quantization noise. These different quantization noise sources are propagated in the system and lead to an output quantization noise. The power of this quantization noise is determined to compute the signal to quantization noise ratio (SQNR). The knowledge of the analytical expression of the SQNR allows to determine the system fixed-point specification for a given SQNR minimal value. Some different models have been proposed for the LMS algorithm as in [6] but no model have been proposed for the APA algorithm.

So, the aim of this paper is to find an analytical expression of the noise power in the APA algorithm for all types of quantization (rounding, convergent rounding and truncation). In convergent rounding, the mean of a noise is equal to zero which is not valid for quantization by rounding and truncation [1]. In section 2, the fixed-point APA algorithm is described and its output is analytically determined in section 3. The model developed is applied to the NLMS with Orthogonal Correction Factors algorithm (NLMS-OCF) in section 4. To finish, in section 5, the accuracy of the model is evaluated by simulations.

2. FIXED-POINT IMPLEMENTATION

The infinite precision APA algorithm can be described as follows

$$\begin{aligned} e_n &= y_n - X_n^t w_n && (1)\\ w_{n+1} &= w_n + \mu X_n [X_n^t X_n + \delta I_K]^{-1} e_n && (2) \end{aligned}$$

where x_n represents the N size vector input data $[x(n), x(n-1), \ldots x(n-N+1)]^t$. Let denote X_n the matrix of K last observation vectors $X_n = [x_n, x_{n-1}, \ldots x_{n-K+1}]$. Thus X_n is a NxK matrix. y_n and e_n are K-tap vectors. δ is a constant used to regularize the matrix $X_n^t X_n$ and I_K the K size identity matrix. The fixed-point model of the APA algorithm is represented on figure 1. The noise terms must be introduced. The regularization term δ is supposed to be a sum of power of 2. The equations of the APA algorithm become :

$$\begin{aligned} e'_n &= y'_n - X'^t_n w'_n - \eta_n && (3)\\ w'_{n+1} &= w'_n + \mu X'_n [X'^t_n X'_n + \delta I_K]^{-1} e'_n + \gamma_n && (4) \end{aligned}$$

where the prime refers to quantified data. γ_n is a N vector white-noise due to the computation of $X'_n[X'^t_n X'_n + \delta I_K]^{-1}$ and e'_n, and is the sum of K multiplication noises. The fixed-point APA is described by the following set of equations :

$$\begin{aligned} X'_n &= X_n + \alpha_n && (5)\\ y'_n &= y_n + \beta_n && (6)\\ [X'^t_n X'_n + \delta I_K]^{-1} &= [X_n^t X_n + \delta I_K]^{-1} + \nu_n && (7)\\ w'_n &= w_n + \rho_n && (8) \end{aligned}$$

with α_n a NxK matrix, β_n and ρ_n a N size vector. Moreover, ν_n is a NxK matrix corresponding to the difference between $[X'^t_n X'_n]^{-1}$ and $[X_n^t X_n]^{-1}$. As demonstrated in [2], ν_n is equal to

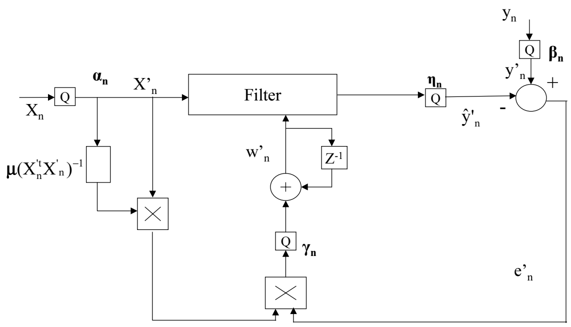

FIG. 1 –. Scheme of the fixed-point APA algorithm

$$\nu_n = [X_n^t X_n + \delta I_K]^{-2}(\alpha_n^t X_n + X_n^t \alpha_n) + \upsilon_n \quad (9)$$

where υ_n is a NxK matrix due to computation of $[X_n'^t X_n']^{-1}$ and I_K the K size identity matrix.

3. NOISE POWER

The aim of this section is the computation of the output noise power. This power is equal to

$$\begin{aligned} E(b_y^2) &= E((X_n'^t w_n' + \eta_n - X_n^t w_n)^2) \\ &= E((\alpha_n^t w_n)^2) + E((X_n^t \rho_n)^2) + E(\eta_n^2) \end{aligned} \quad (10)$$

3.1. Input noise

The first noise term in equation 10 is given by the propagation of the input noise α_n through the system.

$$b_{x_n} = \alpha_n^t w_n \quad (11)$$

b_{x_n} is a K-tap vector. Let B_{x_n} be the correlation matrix of the vector b_{x_n}.

$$B_{x_n} = E(b_{x_n} b_{x_n}^t) \quad (12)$$

To determine the noise power, the matrix B_{x_n} is calculated. With the trace operator properties, the term $Tr(A^t BA)$ is equal to $Tr(AA^t B)$.

So, by denoting, $W_n = E(w_n w_n^t)$ and using the non-correlation between the coefficient w_n and the input noise α_n, the trace of B_{x_n} is equal to

$$Tr(B_{x_n}) = Tr(E(\alpha_n \alpha_n^t) W_n) \quad (13)$$

where $E(\alpha_n \alpha_n^t)$ is a N size square matrix equal to

$$E(\alpha_n \alpha_n^t) = K\sigma_\alpha^2 I_N + K m_\alpha^2 1_N \quad (14)$$

with m_α and σ_α^2 the mean and the variance of α_n, and 1_N the unitary matrix of size N. Thus, equation (13) becomes

$$Tr(B_{x_n}) = K\sigma_\alpha^2 Tr(W_n) + K m_\alpha^2 Tr(1_N W_n) \quad (15)$$

Recognizing that, at convergence steady-state w_n is equal to the optimum coefficient vector w_{opt}, we can write $Tr(W_n) = \sum_{i=0}^{N-1} w_{opt_i}^2$ and $Tr(1_n W_n) = (\sum_{i=0}^{N-1} w_{opt_i})^2$

Thus, the noise term B_{x_n} can be described as follows

$$Tr(B_{x_n}) = K\sigma_\alpha^2 \sum_{i=0}^{N-1} w_{opt_i}^2 + K m_\alpha^2 (\sum_{i=0}^{N-1} w_{opt_i})^2 \quad (16)$$

3.2. Coefficients noise

The noise due to coefficients corresponding to the second term of equation (10) is given by

$$b_{w_n} = X_n^t \rho_n \quad (17)$$

So the correlation matrix of b_{w_n} is equal to

$$B_{w_n} = E(X_n^t \rho_n \rho_n^t X_n) \quad (18)$$

The power of this term is the trace of B_{w_n} which is equal to

$$Tr(B_{w_n}) = Tr(E(X_n X_n^t) E(\rho_n \rho_n^t)) \quad (19)$$

From the equations (1), (3) and (5), the next recursion can be written

$$\rho_{n+1} = F_n \rho_n + b_n \quad (20)$$

where

$$F_n = I_N - \mu X_n [X_n^t X_n]^{-1} X_n^t \quad (21)$$

and

$$\begin{aligned} b_n \;\; &= \underbrace{\gamma_n}_{b_{1_n}} + \underbrace{\mu X_n [X_n^t X_n]^{-1}(\beta_n - \eta_n)}_{b_{2_n}} \\ &+ \underbrace{\mu \alpha_n [X_n^t X_n]^{-1} e_n}_{b_{3_n}} + \underbrace{\mu X_n \nu_n e_n}_{b_{4_n}} \\ &- \underbrace{\mu X_n [X_n^t X_n]^{-1} \alpha_n^t w_n}_{b_{5_n}} \end{aligned} \quad (22)$$

The terms ρ_n, F_n and b_n are considered non-correlated. Denoting the correlation matrix $P_n = E(\rho_n \rho_n^t)$, equation (20) leads to

$$\begin{aligned} P_{n+1} &= P_n E(F_n F_n^t) + E(b_n b_n^t) \\ &+ E(F_n)E(\rho_n)E(b_n^t) + E(b_n)E(\rho_n^t)E(F_n^t) \end{aligned} \quad (23)$$

So, denoting $B_n = E(b_n b_n^t)$ and, at the steady-state, $P_\infty = \lim_{n \longrightarrow \infty} P_n$

$$\begin{aligned} E(X_n X_n^t) P_\infty &= E(X_n X_n^t)(I_N - E(F_n F_n^t))^{-1} [B_n \\ &+ E(F_N)(I_N - E(F_N))^{-1} E(b_n) E(b_n^t) \\ &+ E(b_n) E(b_n^t)(I_N - E(F_N))^{-1} E(F_N)] \end{aligned} \quad (24)$$

Thus, the power of the noise due to coefficients quantization is obtained by using the trace of equation (24). The terms $E(F_n F_n^t)$, $E(F_n)$ and $E(X_n X_n^t)$ depend on the input signal properties and can be easily obtained by simulation. The other terms can be developed to split them up into signal and noise terms.

3.3. Noise due to filter computation

The noise due to filter computation η_n is a K-tap vector. Its correlation matrix $E(\eta_n \eta_n^t)$ is given by

$$E(\eta_n \eta_n^t) = \begin{pmatrix} m_\eta^2 + \sigma_\eta^2 & m_\eta^2 & \dots & m_\eta^2 \\ m_\eta^2 & m_\eta^2 + \sigma_\eta^2 & \dots & m_\eta^2 \\ \dots & \dots & \dots & \dots \\ m_\eta^2 & m_\eta^2 & \dots & m_\eta^2 + \sigma_\eta^2 \end{pmatrix}$$

where m_η is the mean of η_n and σ_η^2 its variance. The power of this term is the trace of its correlation matrix

$$Tr(E(\eta_n \eta_n^t)) = K(m_\eta^2 + \sigma_\eta^2) \tag{25}$$

3.4. Noise power

Using equations (16), (24) and (25), the global noise power on the system output is

$$\begin{aligned} B_y = {} & K(\sigma_\alpha^2 \sum_{i=0}^{N-1} w_{opt_i}^2 + m_\alpha^2 (\sum_{i=0}^{N-1} w_{opt_i})^2 + m_\eta^2 + \sigma_\eta^2) \\ & + Tr(E(X_n X_n^t)(I_N - E(F_n F_n^t))^{-1}[B_n \\ & + E(F_N)(I_N - E(F_N))^{-1} E(b_n) E(b_n^t) \\ & + E(b_n) E(b_n^t)(I_N - E(F_N))^{-1} E(F_N)]) \end{aligned} \tag{26}$$

To develop totally the terms $E(b_n)$, B_n and $E(F_n F_n)$, the type of used APA algorithm must be known. Thus, in the next part, the equation (26) is developed according to the NLMS-OCF algorithm which can be considered as an APA algorithm.

4. CASE OF THE NLMS-OCF

The aim of this section is to determine the noise power in the case of the NLMS-OCF algorithm [4] and to compare with the NLMS algorithm. The NLMS-OCF algorithm is an APA algorithm in which the K last observation vectors are orthogonal. The idea is to simplify the expression of the noise power in this case. To simplify the expressions, x_n is supposed to have zero-mean.

The term $E(X_n (X_n^t X_n)^{-1} X_n^t)$ can be approximated by

$$E(X_n (X_n^t X_n)^{-1} X_n^t) \approx \frac{K}{N} I_N \tag{27}$$

4.1. Coefficients noise in NLMS-OCF

Each term of the coefficient noise expression presented in equation (24) can be simplified in the case of NLMS-OCF

$$(I_N - E(F_n F_n^t)) \approx (2\mu - \mu^2) \frac{K}{N} I_N \tag{28}$$

$$E(X_n X_n^t) \approx K \sigma_x^2 I_N \tag{29}$$

$$(I_N - E(F_n)) \approx \mu \frac{K}{N} I_N \tag{30}$$

Using equation (24), the noise due to the coefficients can be written as

$$\begin{aligned} & Tr(E(X_n X_n^t) P_\infty) \\ & = \frac{N \sigma_x^2}{(2\mu - \mu^2)} [Tr(B_n) + 2 \frac{N(1 - \mu \frac{K}{N})}{K\mu} Tr(E(b_n) E(b_n^t))] \end{aligned} \tag{31}$$

So, the terms $Tr(B_n)$ and $Tr(E(b_n)E(b_n^t))$ must be computed to determine completely the noise due to coefficients. Denoting B_{i_n} the autocorrelation matrix of each of the five terms b_{i_n} of the expression (22), and supposing that they are not correlated, the next equality is obtained

$$Tr(B_n) = \sum_{i=1}^{5} Tr(B_{i_n}) \tag{32}$$

In the following lines, each term B_{i_n} is developed

$$Tr(B_{1_n}) = Tr(E(\gamma_n \gamma_n^t)) = K^2 N m_\gamma^2 + K N \sigma_\gamma^2 \tag{33}$$

Here, b_1 can be considered as a sum of K uncorrelated noises with the same probability density. So, its power is equal to the power in the case of the NLMS algorithm multiplied by K.

$$\begin{aligned} Tr(B_{2_n}) = {} & \mu^2 Tr[((\sigma_\beta^2 + \sigma_\eta^2) I_N + (m_\beta^2 + m_\eta^2) 1_N) \times \\ & (E(X_n (X_n^t X_n)^{-2} X_n^t))] \\ & = K\mu^2 (E(\beta^2) + E(\eta^2)) \frac{1}{N\phi_x + N(N-1)\sigma_x^4} \end{aligned} \tag{34}$$

where ϕ_x is the kurtosis of the input signal x_n. As for b_1, b_2 can be expressed by the sum of K noise terms. Its power is only composed by its variance because its mean is equal to 0 since $m_x = 0$. To determine $Tr(B_{3_n})$, e_n is supposed to have zero-mean and a variance ξ.

$$\begin{aligned} Tr(B_{3_n}) = {} & \mu^2 Tr[(\sigma_\alpha^2 I_N + m_\alpha^2 1_N) \\ & (E((X_n^t X_n)^{-1} e_n e_n^t (X_n^t X_n)^{-1}))] \\ & = K\mu^2 (\sigma_\alpha^2 + m_\alpha^2) \frac{\xi}{N\phi_x + N(N-1)\sigma_x^4} \end{aligned} \tag{35}$$

b_3 is also the sum of K noise terms but with zero-mean since e_n has zero-mean. The two other terms B_{4_n} and B_{5_n} can be developed in the same manner. So compared to the NLMS algorithm, $Tr(B_n)$ is multiplied by K.

To determine completely the coefficients noise in the NLMS-OCF algorithm, $Tr(E(b_n)E(b_n^t))$ must be developed. The term $E(b_n)$ is equal to

$$E(b_n) = [K m_\gamma, K m_\gamma, \dots K m_\gamma]^t \tag{36}$$

So the term $Tr(E(b_n)E(b_n^t))$ is equal to

$$Tr(E(b_n)E(b_n^t)) = N K^2 m_\gamma^2 \tag{37}$$

This term can also be interpreted as the sum of K noise sources. But, it is multiplied by K^2 since it represents the square of the mean. Thus, each noise term in the NLMS-OCF (b_1, b_2, b_3, b_4 and b_5) is a sum of K terms whereas in the NLMS they represent only one term. So the power of the coefficient noise is multiplied by K in the NLMS-OCF.

4.2. Data noise in the NLMS-OCF

In the NLMS-OCF, as in equation (16), the power of the data noise is equal to

$$Tr(E(\alpha_n \alpha_n^t)W_n) = K\sigma_\alpha^2 \sum_{i=0}^{N-1} w_{opt_i}^2 + Km_\alpha^2 (\sum_{i=0}^{N-1} w_{opt_i})^2 \tag{38}$$

In the NLMS, the power of the data noise is [6]

$$Tr(E(\alpha_n \alpha_n^t)W_n) = \sigma_\alpha^2 \sum_{i=0}^{N-1} w_{opt_i}^2 + m_\alpha^2 (\sum_{i=0}^{N-1} w_{opt_i})^2 \tag{39}$$

So, in the NLMS-OCF case, the power of data noise is multiplied by K

4.3. Computation filter noise in the NLMS-OCF

As in equation (25), the power of the computation filter noise is

$$Tr(E(\eta_n \eta_n^t)) = K(m_\eta^2 + \sigma_\eta^2) \tag{40}$$

In the NLMS case, it is equal to [6]

$$Tr(E(\eta_n \eta_n^t)) = m_\eta^2 + \sigma_\eta^2 \tag{41}$$

So, in the NLMS-OCF case, the power is multiplied by K. The global noise power in the NLMS-OCF is equal to the global noise power in the NLMS algorithm multiplied by K. So, the NLMS-OCF is a K-NLMS in terms of noise

5. RESULTS

In this part, the quality of this model to estimate $E(b_y^2)$ is evaluated. For these experiments, tests are made for quantization by truncation and by rounding. The relative error between the noise power obtained with fixed-point simulations and the estimated noise power with our model described in equation (26) is computed. The chosen input signal is an autoregressive process in which the correlation coefficient β between data input can be fixed between 0 (white-noise) and 1 (very correlated).

5.1. Results for the APA algorithm in truncation

The figure 2 shows the accuracy of the model for N between 1 and 20, K between 1 and N and for very correlated input data ($\beta = 0.9$). This relative error is smaller than 30% which is a good result since it represents a difference of only 2 dB between the output quantization noise power estimated by simulation and the power given by our model. So, this new developed model is valid for the case of quantization by truncation and for very correlated inputs. For less correlated inputs, good results are also obtained.

5.2. Results for the NLMS-OCF algorithm in rounding

The simulations are made for orthogonal input data in the case of quantization by rounding and the results are in figure 3. The parameters are the same as in section 5.1. As in the truncation quantization case, our model leads to an accurate estimation of the noise power. The relative error is smaller than 30%.

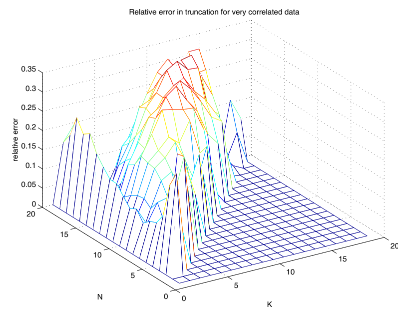

FIG. 2 –. Relative error for very correlated input data with quantization by truncation

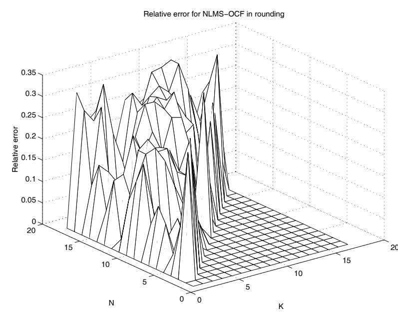

FIG. 3 –. Relative error for orthogonal input data with quantization by rounding

6. CONCLUSION

In this paper, a model to determine analytically the noise power output of the APA algorithm is presented. The model is developed for the general algorithm and after, is simplified for the NLMS-OCF algorithm. The simulation results show that our model is accurate. This approach has for main advantage to be valid for all types of quantization. This methodology, which has been successfully used for LMS and NLMS algorithms, is spread to APA algorithms. Nevertheless, further studies have to be carried out in order to generalize this approach for all types of systems and particularly, non-linear systems.

7. REFERENCES

[1] G. Constantinides, P. Cheung and W. Luk "Truncation Noise in Fixed-Point SFGs", *IEE Electronic Letters, 35(23), pp: 2012-2014, November 1999.*

[2] D.Menard, R. Rocher, P. Scalart and O.Sentieys , "Automatic SQNR determination in non-linear and non-recursive fixed-point systems", *EUSIPCO, pp:1349-1352, September 2004.*

[3] K. Ozeki and T.Umeda, "An adaptative filtering algorithm using an orthogonal projection to an affine subspace and its properties", *Electr. Commun. Jpn., vol. 67, pp: 19-27, 1984.*

[4] S.D. Sankaran and A.A. Beex , "Convergence Behavior of Affine Projection Algorithms", *IEEE Transactions Acoustic, Speech, Signal Processing, vol ASSP-48, no.4, pp: 1086-1096, April 2000.*

[5] A.H. Sayed and H.C. Shin "Transient behavior of Affine Projections Algorithms", *ICASSP, vol: 52 , pp:90 - 102, January 2004*

[6] R. Rocher, D. Menard, O. Sentieys and P. Scalart , "Accuracy Evaluation of Fixed-Point LMS", *ICASSP, vol: 5, pp: 237 - 240 May 2004.*