ON THE PROBABILITY OF RESOLUTION FOR THE AMPLITUDE AND PHASE ESTIMATION (APES) SPECTRAL ESTIMATOR

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ABSTRACT

The Amplitude and Phase EStimation (APES) algorithm is a spectral estimation approach that estimates the complex amplitude of the power spectrum of a random process. Although its resolution performance has been observed to be slightly better than conventional FFT approaches, but quite inferior to super-resolution approaches like the Capon algorithm and MUSIC, no quantitative measure of resolution exists for the APES algorithm. This analysis provides a new exact two point measure of the large sample probability of resolution for the APES algorithm, as well as an approximation useful in capturing finite sample effects. This probability measure indicates that the APES algorithm resolution performance is fundamentally limited even in the limit of infinite signal-to-noise ratio (SNR).

1. INTRODUCTION

The Amplitude and Phase EStimation (APES) algorithm can be interpreted as a finite impulse response (FIR) filtering approach to spectral estimation. It focuses on complex amplitude estimation and can be derived via several optimization criteria, among which is a maximum-likelihood (ML) formulation [7, 6]. If each of the data snapshots is assumed independent identically complex Gaussian distributed, then the APES spectrum corresponds to the ML estimate of the data spectral complex amplitude. It is straightforward to continue with the ML formulation of APES, however, to obtain an optimal estimate of the frequency of a single sinusoid in colored noise. This leads to a well-known form of ML signal parameter estimation requiring the intermediate estimation of a colored noise covariance. This form appears quite often in adaptive array applications (see [8] and refs. therein). The primary difference between the ML frequency estimator and the ML complex amplitude estimator (or APES) lies in the filter weight normalization. This present analysis defines a new two point measure of the probability of resolution for the APES algorithm (*i.e.* the ML complex amplitude estimator), and the ML frequency estimator. The large sample limiting performance differences resulting from the differing weight normalizations are explored herein via exact new closed form expressions for the two point probability of resolution for both the APES algorithm and the ML frequency estimator, both accounting for signal model mismatch [3] and colored noise. The additional impact of finite sample effects [2, 9, 5] on resolution are explored for the ML frequency estimator from which a useful approximation for the APES probability follows.

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2. THE APES ALGORITHM

The goal of spectral estimation is to obtain from a finite set of data measurements an estimate of the power distribution over frequency of a random process. This present analysis shall focus on direction of arrival (DOA) estimation with uniform linear arrays (ULAs), although applicable to a broad class of problems. Thus, the power distribution over spatial frequency, or angle of arrival, is of interest, and data observations are taken in space across an array of sensors. Let the set of data observations (snapshots) taken from an N element array for L looks be denoted collectively by the $N \times L$ data matrix $\mathbf{X} = [\mathbf{x}(1)|\mathbf{x}(2)|\cdots|\mathbf{x}(L)]$. Each snapshot is assumed complex Gaussian distributed such that $\mathbf{x}(l) \sim \mathcal{CN}_N[S(\theta_T)e^{j\omega(l-1)}\mathbf{v}(\theta_T), \mathbf{R}]$ for $l = 1, 2, \ldots, L$ where the signal angle of arrival is θ_T, and the array response is $\mathbf{v}(\theta_T)$, and the complex signal amplitude is given by $S(\theta_T)$, and the known relative phase progression from look-to-look is $e^{j\omega(l-1)}$ (*e.g.* Doppler phase in radar), and $\mathbf{R}$ is the colored noise covariance. The APES algorithm chooses filter $\mathbf{w}$ and complex amplitude $S(\theta)$ as joint minimizers of the following criterion:

$$\min_{\mathbf{w},S} \sum_{l=1}^{L} \left| \mathbf{w}^H \mathbf{x}(l) - S(\theta) e^{j\omega(l-1)} \right|^2 . \tag{1}$$

The APES complex amplitude spectral estimate at angle θ is given by

$$\widehat{S}_{APES}(\theta) = \frac{\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\boldsymbol{\mu}}{\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\mathbf{v}(\theta)} \triangleq \widehat{\mathbf{w}}^H_{APES}(\theta)\boldsymbol{\mu} \tag{2}$$

where $\boldsymbol{\mu} \triangleq \frac{1}{L}\mathbf{X}\mathbf{h}$, and $\mathbf{h} = [1, e^{-j\omega}, \ldots, e^{-j\omega(L-1)}]^T$, and $\widehat{\mathbf{R}} \triangleq \mathbf{X}\mathbf{X}^H - L\boldsymbol{\mu}\boldsymbol{\mu}^H$, and the filter weight $\widehat{\mathbf{w}}_{APES}(\theta)$ has been defined. This spectral estimate likewise follows from a ML formulation of the problem [7, 6]. Further maximization of the likelihood function results in the following estimate of the spatial frequency or angle of arrival θ_T of a single signal:

$$\widehat{\theta}_T = \arg\max_{\theta} \frac{\left|\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\boldsymbol{\mu}\right|^2}{\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\mathbf{v}(\theta)} \triangleq \left| L\widehat{\mathbf{w}}^H_{FE}(\theta)\boldsymbol{\mu}\right|^2 \tag{3}$$

where the filter weight used for frequency estimation $\widehat{\mathbf{w}}_{FE}(\theta)$ has been defined. This frequency estimate is well-known and widely used [8]. The large sample filter weights for APES and frequency estimation (FE) are respectively denoted as

$$\mathbf{w}_{APES}(\theta) = \frac{\mathbf{R}^{-1}\mathbf{v}(\theta)}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}, \quad \mathbf{w}_{FE}(\theta) = \frac{\mathbf{R}^{-1}\mathbf{v}(\theta)}{\sqrt{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}} \tag{4}$$

and the APES large sample spectral estimate as $S_{APES}(\theta) \triangleq \mathbf{w}^H_{APES}(\theta)\boldsymbol{\mu}$. One filter is asymptotically efficient for complex amplitude estimation [10] while the other is asymptotically efficient for frequency estimation of a single sinusoid in a colored noise background [11].

Under the complex Gaussian data assumptions made herein, it can be proven that $\boldsymbol{\mu}$ and $\widehat{\mathbf{R}}$ are statistically independent and further that $\boldsymbol{\mu}$ is complex Gaussian distributed and $\widehat{\mathbf{R}}$ has a central complex Wishart distribution [11]. Thus, the exact probability density function (pdf) for the APES complex amplitude estimate is known [10], and accurate prediction of the mean squared error performance of the ML frequency estimator non-asymptotically is possible [11]. The APES algorithm, however, can be used in an ad hoc manner to obtain estimates of the location of spectral line components [4, 6], *i.e.* to estimate the frequencies of pure sinusoids (planewaves in space). Refs. [4, 6] show that the APES resolution capability is slightly better than conventional FFT processing, but inferior to super-resolution approaches like the Capon algorithm [1, 12] and MUSIC. It is desired in this analysis to provide a useful quantitative statistical measure for the limiting large sample resolution performance of the APES spectral estimator accounting for signal mismatch and colored noise. In addition a useful approximation is obtained for the APES and FE finite sample resolution performance.[1]

3. SPECTRAL RESOLUTION AND THE APES AMBIGUITY FUNCTION

The ability of an algorithm to resolve two closely spaced sources is often measure by quantifying the likelihood of a "dip" appearing in the estimated spectrum between the two sources. The ambiguity function provides much insight into the APES algorithm's resolving capacity. Let the mean of each snapshot containing two closely spaced sources be given by

$$\mathbf{d} \triangleq \gamma_0\mathbf{v}(\theta_0) + \gamma_1\mathbf{v}(\theta_0 + \delta\theta) \tag{5}$$

where $\delta\theta$ is the angle separation and parameters γ_0 and γ_1 can be used to adjust SNR levels. The natural definitions of the APES ambiguity function is given by:

$$\psi_{APES}(\theta) \triangleq \left|\mathbf{w}^H_{APES}\mathbf{d}\right|^2 = \frac{\left|\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{d}\right|^2}{\left[\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)\right]^2}. \tag{6}$$

Although the concept of resolution lacks a rigorous definition, the widely accepted rule is the following: *If $\widehat{S}(\theta)$ is the complex spectral estimate obtained from an algorithm, then the two signals are said to be resolved if*

$$\left|\widehat{S}(\theta_0 + \delta\theta/2)\right| < \alpha \cdot \left(\left|\widehat{S}(\theta_0)\right| + \left|\widehat{S}(\theta_0 + \delta\theta)\right|\right) \tag{7}$$

where parameter α defines the amount of dip; e.g. $\alpha = \sqrt{0.5}$ for equal power sources indicates at least a 3dB dip in power between these signals. Since the estimated spectrum is a stochastic process, some approximation of the probability of this three point event would be most useful. Note from the ambiguity function that resolution performance is akin to analysis in the presence of signal model mismatch [3, 5].

4. TWO POINT MEASURE OF THE APES ALGORITHM PROBABILITY OF RESOLUTION

A useful two point measure of the probability of resolution providing accurate prediction of the SNR at which sources can be resolved by the APES algorithm can be defined. Let the snapshots be distributed as $\mathbf{x}(l) \sim \mathcal{CN}_N(\mathbf{d}, \mathbf{R})$ where the data mean is given by equation (5). Define parameter θ_{MP} as the parameter value of the source with the smallest power out of the ambiguity function, *i.e.* $\theta_{MP} \triangleq \arg \min_{\theta_0, \theta_0+\delta\theta} \psi_{APES}(\theta)$. The large sample APES probability of resolution can be defined as

$$P^{APES}_{res}(\theta_0, \theta_0 + \delta\theta) \triangleq \Pr\left[\left|S_{APES}\left(\theta_0 + \delta\theta/2\right)\right| < \sqrt{\alpha} \cdot \left|S_{APES}(\theta_{MP})\right|\right]. \tag{8}$$

Similarly, the large sample probability of resolution for the ad hoc approach of using the single signal ML frequency estimator to obtain estimates of multiple signals is given by

$$P^{FE}_{res}(\theta_0, \theta_0 + \delta\theta) \triangleq \Pr\left[\left|\mathbf{w}^H_{FE}\left(\theta_0 + \delta\theta/2\right)\boldsymbol{\mu}\right| < \sqrt{\alpha} \cdot \left|\mathbf{w}^H_{FE}(\theta_{MP})\boldsymbol{\mu}\right|\right]. \tag{9}$$

Since perfect knowledge of the data covariance is assumed, random variations essentially derive from the presence of $\boldsymbol{\mu}$, which is primarily a function of the SNRs resulting from the choice of γ_0 and γ_1. These probabilities have been derived in closed form in [11]. The next subsections summarize the algorithms for computing these probabilities.

4.1. Large Sample APES Resolution Probability

A complex non-central chi-squared random variable of M complex degrees of freedom and non-centrality parameter δ will be denoted by $\chi^2_M(\delta)$. Define the function $\mathcal{J}_e$ as follows:

$$\mathcal{J}_e(\delta_1, \delta_2, h) \triangleq \Pr\left(\frac{{}_1\chi^2_1(\delta_1)}{{}_2\chi^2_1(\delta_2)} > h\right) = \left(\frac{h}{1+h}\right) \times \left[1 + \frac{1}{h}Q_1\left(\frac{h\delta_2}{1+h}, \frac{\delta_1}{1+h}\right) - Q_1\left(\frac{\delta_1}{1+h}, \frac{h\delta_2}{1+h}\right)\right] \tag{10}$$

where the generalized Marcum-Q-function is given by

$$Q_M(u, \delta) \triangleq \Pr\left(\chi^2_M(\delta) > u\right) = \int_u^\infty \left(\frac{a}{\delta}\right)^{\frac{(M-1)}{2}} e^{-(a+\delta)} I_{M-1}(2\sqrt{\delta a})da \tag{11}$$

[1] The ML frequency estimator described is that which assume only a single signal is present in the data. If multiple signals are present, then the ML estimator takes on a different form requiring joint estimation of the multiple frequencies. Often the number of signals present is unknown. Thus, it is common to take the ad hoc approach of scanning the search space over frequency and look for peaks to identify the presence of a signal [12].

and $I_{M-1}(x)$ is the modified Bessel function of the first kind of order $M-1$. Define the following function $\mathcal{P}_e$ in terms of $\mathcal{J}_e$:

$$\mathcal{P}_e(\delta_1,\delta_2,\lambda_1,\lambda_2) \triangleq \Pr\left[{}_1\chi_1^2(\delta_1)\lambda_1 + {}_2\chi_1^2(\delta_2)\lambda_2 > 0\right]$$
$$= \begin{cases} \mathcal{J}_e(\delta_1,\delta_2,l_\lambda), & \text{sign}(\lambda_1)=1,\ \ \lambda_1,\lambda_2\neq 0 \\ 1-\mathcal{J}_e(\delta_1,\delta_2,l_\lambda), & \text{sign}(\lambda_1)=-1,\ \ \lambda_1,\lambda_2\neq 0 \\ 1, & \text{sign}(\lambda_1)=1,\ \ \lambda_2=0 \\ 0, & \text{sign}(\lambda_1)=-1,\ \ \lambda_2=0 \end{cases} \quad (12)$$

where $l_\lambda \triangleq -\lambda_2/\lambda_1$.

The algorithm for computation of the APES algorithm large sample probability of resolution is as follows:

1. Let $\theta_{MP}=\theta_1$ and $\theta_0+\delta\theta/2=\theta_2$. Define the following matrices: $\mathbf{V}=[\mathbf{v}(\theta_1)|\mathbf{v}(\theta_2)]$, and $\mathbf{A}_{APES}$ a diagonal matrix with components $[\mathbf{A}_{APES}]_{k,k}=1/\mathbf{v}^H(\theta_k)\mathbf{R}^{-1}\mathbf{v}(\theta_k)$, $k=1,2$. Form the matrix of two point weight vectors $\mathbf{W}_{APES}=\mathbf{R}^{-1}\mathbf{V}\mathbf{A}_{APES}$.
2. Compute the covariance $\mathbf{R}_{WX}\triangleq\mathbf{W}_{APES}^H\mathbf{R}\mathbf{W}_{APES}$ and its matrix square root $\mathbf{R}_{WX}^{1/2}$.
3. Form the following 2×2 matrix and eigen-decompose:
$$\mathbf{R}_{WX}^{1/2}\begin{bmatrix}\alpha & 0\\ 0 & -1\end{bmatrix}\mathbf{R}_{WX}^{1/2}=\mathbf{Q}_{WX}^H\mathbf{\Lambda}_{WX}\mathbf{Q}_{WX}. \quad (13)$$
4. Choose the desired signal levels via choice of γ_0 and γ_1. Choose $\mathbf{d}$ as in (5) and form the following 2×1 complex vector
$$\boldsymbol{\rho}_{SNR}=\sqrt{L}\cdot\mathbf{Q}_{WX}\mathbf{R}_{WX}^{-1/2}\mathbf{W}_{APES}^H\mathbf{d}.$$
5. Use the magnitude squared of the elements of $\boldsymbol{\rho}_{SNR}$ and the eigenvalues $\mathbf{\Lambda}_{WX}$ to compute the desired pairwise error probability via

$$P_{res}^{APES}(\theta_0,\theta_0+\delta\theta)= \mathcal{P}_e\left(|\rho_{SNR,1}|^2,|\rho_{SNR,2}|^2,\lambda_{WX,1},\lambda_{WX,2}\right). \quad (14)$$

The algorithm for computing $P_{res}^{FE}(\theta_0,\theta_0+\delta\theta)$ the resolution probability for the FE approach is exactly the same as the APES algorithm with the minor change that the diagonal entries of the diagonal matrix $\mathbf{A}$ in step 1) be replaced by

$$[\mathbf{A}_{FE}]_{k,k}=1/\sqrt{\mathbf{v}^H(\theta_k)\mathbf{R}^{-1}\mathbf{v}(\theta_k)},\quad k=1,2. \quad (15)$$

4.2. Fundamental Limit on APES Algorithm Resolution

Analysis of these resolution probabilities indicates that *the APES resolution performance is fundamentally limited by the smallest nonzero angle (or frequency) seperation $\delta\theta$ satisfying the condition that*

$$|\rho_{SNR,1}|=|\rho_{SNR,2}|, \quad (16)$$

which for the white noise case can be expressed in terms of the ambiguity function as the smallest nonzero $\delta\theta$ such that:

$$\psi_{APES}(\theta)=\psi_{APES}\left(\theta+\frac{\delta\theta}{2}\right) \quad (17)$$

even in the limit of infinite SNR. Such a limit on resolution performance is well-known for the conventional FFT approach [12], but is somewhat surprising for this adaptive approach since Capon and MUSIC resolution improves with SNR. The APES resolution limit, however, can at times modestly exceed the Fourier limit [4] (better resolution is predicted at times when sources are of unequal power).

4.3. On Finite Sample Effects

The previous section assumed perfect knowledge of the data covariance parameter $\mathbf{R}$ and explored resolution exclusively as a function of SNR. Finite sample effects will impact the SNR and the fidelity of its estimate $\widehat{\mathbf{R}}$. Thus, it is desired to understand the combined impact of finite sample size L on estimates $\boldsymbol{\mu}$ and $\widehat{\mathbf{R}}$ with regards to resolution. To this end, the two point measure for the ML FE is defined:

$$\tilde{P}_{res}^{FE}(\theta_0,\theta_0+\delta\theta)\triangleq \Pr\left[\left|\widehat{\mathbf{w}}_{FE}^H(\theta_0+\delta\theta/2)\,\boldsymbol{\mu}\right|<\sqrt{\alpha}\cdot\left|\widehat{\mathbf{w}}_{FE}^H(\theta_{MP})\boldsymbol{\mu}\right|\right]. \quad (18)$$

It can be shown [11] that the exact probability is given by an expression of the form

$$\tilde{P}_{res}^{FE}(\theta_0,\theta_0+\delta\theta)=\int_0^1 d\beta\cdot P_{\beta_{L-N+3,N-2}}(\beta;\delta_{\mathbf{V}_\perp}) \times\int_{-\infty}^0 dF\cdot P_{F_\Delta}(F;\mathbf{\Delta}) \mathcal{P}_e\left[\beta|\tilde{\rho}_{SNR,1}(F)|^2,\beta|\tilde{\rho}_{SNR,2}(F)|^2,\tilde{\lambda}_{WX,1}(F),\tilde{\lambda}_{WX,2}(F)\right]. \quad (19)$$

where the double integral is with respect to two independent random variables that include a non-central complex beta distribution and a variant on the central complex F distribution. The beta random variable captures the loss in SNR due to covariance estimation [9, 5], as evidenced by its multiplication of the terms governing SNR. The central F statistic captures the statistical dependence among the two adaptive weights due to their reliance upon the same covariance estimate.

An approximation of the finite sample APES resolution probability is possible by recognizing that the beta random variable represents the loss in SNR due to covariance estimation. Modifying the large sample approximation by replacing the SNR variables γ with

$$\tilde{\gamma}_k=E\{\beta_{L-N+3,N-2}(\delta_{\mathbf{V}_\perp})\}\cdot\gamma_k,\quad\text{for } k=0,1 \quad (20)$$

leads to the approximation of the APES finite sample resolution probability. The beta expectation is known in closed form [5, 11] and shown to be less than unity.

5. NUMERICAL EXAMPLES

Consider two closely spaced planewave sources of equal power as observed on an $N=24$ element ULA with a 4.8 degree beamwidth. One signal is placed at array broadside and the other $\delta\theta$ away from broadside. The large sample probability of resolution for the APES and FE algorithms is plotted in Figure 2 (A) as a function of the source separation $\delta\theta$ for various SNRs (element level SNR after averaging L snapshots). The APES and FE resolution performance is nearly identical in white noise and improves with signal strength, but is clearly limited where the limit is given by condition (17) and illustrated in Figure 1. Figure 2 (B) shows the finite sample resolution probability for the FE algorithm when each snapshot has a -20dB element level SNR (dashed lines) alongside its large sample probability having an integrated element level SNR of 0dB. Thus, for low -20dB SNR signals at least $5N$ samples are required to resolve with confidence at the APES/FE resolution limit for this array. Lastly, Figure 2 (C) illustrates the accuracy of the approximation of the finite sample resolution probability obtained via (20) for same array when each snapshot has a -20dB element level SNR. The accuracy of the approximation obviously improves with sample support.

6. CONCLUSIONS

A new two point measure for the probability of resolution for the APES algorithm has been defined and derived in closed form for the large sample case and approximated for the finite sample case. Although empirical studies have been made [4], this present analysis provides the first quantitative measure of resolution performance for the APES algorithm. The resolution probability accounts for signal model mismatch and colored noise. The APES algorithm resolution performance is fundamentally limited even at very large SNRs. This fundamental limitation is reminiscent of conventional approaches known to obey the Fourier/Rayleigh limit [12], although APES can do slightly better. An exact condition for the angle separation corresponding to this resolution limit for APES was obtained, and for white noise expressed in terms of the ambiguity function.

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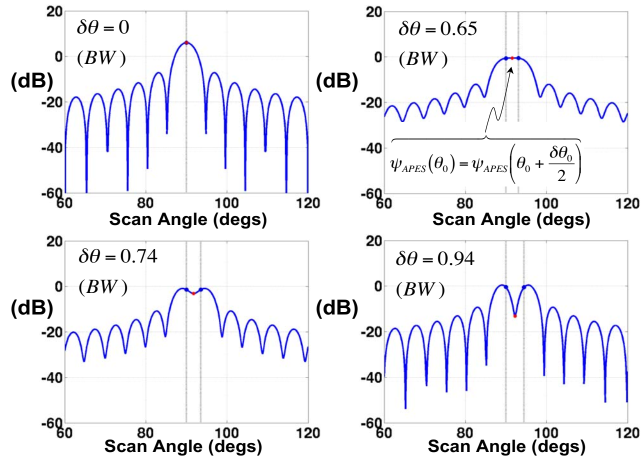

Fig. 1. APES Ambiguity Function for Two Closely Spaced Signal of Equal Strength, $N = 24$ element ULA, $\delta\theta$ in Beamwidths

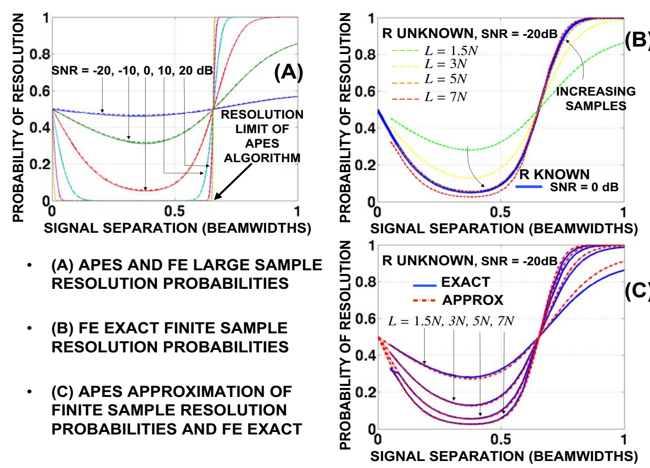

Fig. 2. APES and FE Probability of Resolution for $N = 24$ element ULA

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