EXPLOITING MULTIPLE SHIFT INVARIANCES IN MULTIDIMENSIONAL HARMONIC RETRIEVAL OF DAMPED EXPONENTIALS

Marius Pesavento[†], *Shahram Shahbazpanahi*[*], *Johann F. Böhme*[†], *and Alex B. Gershman*[*]

[†]Faculty of Electrical Engineering and Information Sciences, Ruhr University, Bochum, Germany

[*]Department of Electrical and Computer Engineering, McMaster University, Hamilton, ON, Canada

ABSTRACT

We address the problem of estimating the frequencies and damping factors of a multidimensional signal which consists of several damped complex exponentials. Such a problem is of interest in several applications such as nuclear magnetic resonance spectroscopy where the 2-dimensional (2D) frequencies and damping factors are used to determine the structure of proteins. We herein propose a new algorithm which exploits the multiple-invariance structure that exists in the data model. Unlike search-based parameter estimation techniques (such as D-MUSIC of [5]), which have been developed for multidimensional harmonic retrieval of damped exponentials, our algorithm uses polynomial rooting to efficiently obtain the parameters of interest in a search-free fashion.

1. INTRODUCTION

Numerous parameter estimation problems arising in sensor array processing, mobile communications, parametric MIMO channel estimation and radar signal processing can be formulated as a one- or multi-dimensional (MD) pure (undamped) harmonic retrieval problem. However, there are other important applications such as MD nuclear magnetic resonance (NMR) spectroscopy, where the underlying signal model represents a sum of several damped MD exponentials, and the estimation of both the MD frequencies and damping factors are of special interest because these parameters are used to identify protein structure of a probe.

Several computationally efficient search-free subspace-based methods have recently been formulated for the undamped harmonic retrieval problem that either belong to the class of ESPRIT-type methods [1, 2, 3, 4] or to the class of so-called rooting-based methods [6]. While the ESPRIT-type methods can be naturally generalized to the damped harmonic scenarios [4], this is not the case for the rooting-based methods. However, the ESPRIT-based methods either do not take advantage of the *full* invariance structure contained in the data [1, 2, 4], or lead to computationally more demanding solutions such as multiple-invariance (MI) ESPRIT of [3].

In this paper, we take a general approach to the MD damped harmonic retrieval problem and combine the MI-ESPRIT concept with polynomial rooting in order to retain the benefits of both approaches, i.e. computational efficiency, applicability to undamped and damped harmonic retrieval problem, and exploitation of complete MI structure. Using this approach, a novel high-resolution search-free technique for the problem of interest is proposed. It is proven that our algorithm yields unique solutions under relatively mild conditions. Simulation results illustrate the performance improvements achieved by our algorithm relative to a popular method of [4] which is known to be one of the best techniques developed for harmonic retrieval of damped exponentials.

2. SIGNAL MODEL

For the sake of presentation simplicity, we restrict ourselves to the 2D damped harmonic retrieval problem. Extensions to the MD signal model are immediate. Consider the following 2D signal which is the mixture of P damped exponentials:

$$\boldsymbol{x}_{k,l} = \sum_{p=1}^{P} c_p a_p^{(k-1)} b_p^{(l-1)} \tag{1}$$

with the generators $a_p = e^{\alpha_{1,p}+j\omega_{1,p}}$ and $b_p = e^{\alpha_{2,p}+j\omega_{2,p}}$ ($p = 1, \ldots, P$) characterizing the damped harmonics, see [5] and reference therein. We assume that $\alpha_{1,p}$ and $\alpha_{2,p}$ are the damping factors (i.e., $|\alpha_{1,p}| \leq 1$ and $|\alpha_{2,p}| \leq 1$) of the pth harmonic along the a- and the b-axes, respectively. Similarly, $\omega_{1,p}$ and $\omega_{2,p}$ denote the frequencies of the pth harmonic observed along the a- and the b-axes, and c_p denotes the complex amplitude of this harmonic. The sample supports along the a- and b-axes are given by $k = 1, \ldots, K$ and $l = 1, \ldots, L$.

To obtain a low-rank data model of a sufficiently large dimension, we rearrange the data samples taken along a- and b-axes to form a $(K_1 L_1) \times (K_2 L_2)$ matrix [4, 5]

$$\tilde{\boldsymbol{Y}} = \begin{bmatrix} \boldsymbol{Y}_1 & \boldsymbol{Y}_2 & \cdots & \boldsymbol{Y}_{K_2} \\ \boldsymbol{Y}_2 & \boldsymbol{Y}_3 & \cdots & \boldsymbol{Y}_{K_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{Y}_{K_1} & \boldsymbol{Y}_{K_1+1} & \cdots & \boldsymbol{Y}_{K_2+K_1-1} \end{bmatrix} \tag{2}$$

where

$$\boldsymbol{Y}_n = \begin{bmatrix} \boldsymbol{x}_{n,1} & \boldsymbol{x}_{n,2} & \cdots & \boldsymbol{x}_{n,L_2} \\ \boldsymbol{x}_{n,2} & \boldsymbol{x}_{n,3} & \cdots & \boldsymbol{x}_{n,L_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}_{n,L_1} & \boldsymbol{x}_{n,L_1+1} & \cdots & \boldsymbol{x}_{n,L_2+L_1-1} \end{bmatrix} \tag{3}$$

and the integers K_1, K_2, L_1 and L_2 that satisfy

$$K = K_1 + K_2 + 1; \quad L = L_1 + L_2 + 1 \tag{4}$$

are chosen so that the reassembled data matrix $\tilde{\boldsymbol{Y}}$ becomes as "large" as possible. It is simple to see that, if we choose the integers so that the minimum of $K_1 L_1$ and $K_2 L_2$ is maximized, then the achievable rank of $\tilde{\boldsymbol{Y}}$, and consequently, the maximum number of identifiable signals is maximized. The reassembled data matrix in (2) allows a simple representation [4, 5]

$$\tilde{\boldsymbol{Y}} = \boldsymbol{H}_1 \boldsymbol{C} \boldsymbol{H}_2^T \tag{5}$$

where

$$\boldsymbol{H}_i = \boldsymbol{B}_i \circ \boldsymbol{A}_i, \ \ \boldsymbol{H}_i \in \mathbb{C}^{(K_i L_i)\times P} \tag{6}$$
$$[\boldsymbol{A}_i]_{k,p} = a_p^{(k-1)}, \ \ \boldsymbol{A}_i \in \mathbb{C}^{K_i \times P} \tag{7}$$
$$[\boldsymbol{B}_i]_{l,p} = b_p^{(l-1)}, \ \ \boldsymbol{B}_i \in \mathbb{C}^{L_i \times P} \tag{8}$$
$$\boldsymbol{C} = \mathrm{diag}\{c_1, \ldots, c_P\} \tag{9}$$

for $i = 1, 2$, and "$\circ$" denotes the Khatri-Rao product of two matrices (the column-wise Kronecker product), i.e., $\boldsymbol{E} \circ \boldsymbol{F} = [\boldsymbol{e}_1 \otimes \boldsymbol{f}_1, \boldsymbol{e}_2 \otimes \boldsymbol{f}_2, \ \ldots \ ]$ where $\boldsymbol{e}_k$ and $\boldsymbol{f}_k$ are, respectively, the kth columns of $\boldsymbol{E}$ and $\boldsymbol{F}$, and $\otimes$ denotes the Kronecker product. The singular value decomposition of the reassembled data matrix can be written as

$$\tilde{\boldsymbol{Y}} = \boldsymbol{U}_1 \boldsymbol{D} \boldsymbol{U}_2^T \tag{10}$$

where $\boldsymbol{U}_1 \in \mathbb{C}^{(K_1 L_1)\times P}$ and $\boldsymbol{U}_2 \in \mathbb{C}^{(K_2 L_2)\times P}$.

3. 2D MULTIPLE-INVARIANCE ESPRIT

In what follows, for simplicity of notation we assume that $K_1 L_1 \geq K_2 L_2 \geq P$. The case where $K_1 L_1 < K_2 L_2$ can be handled under the same framework by transposing the reassembled data matrix in (2). The signal matrix $\boldsymbol{H}_1$ represents the Khatri-Rao product of two Vandermond matrices, and, hence, it contains multiple invariant submatrices. Let $\overline{\boldsymbol{A}}_{1,k}$ and $\underline{\boldsymbol{A}}_{1,k}$ denote the $(K_1 - k) \times P$ matrices formed, respectivley, from the first and the last $K_1 - k$ rows of $\boldsymbol{A}_1$ for $k = 1, \ldots, K_1 - 1$, that is,

$$\overline{\boldsymbol{A}}_{1,k} = \overline{\boldsymbol{J}}_{K_1,k} \boldsymbol{A}_1 \tag{11}$$
$$\underline{\boldsymbol{A}}_{1,k} = \underline{\boldsymbol{J}}_{K_1,k} \boldsymbol{A}_1 \tag{12}$$

where $\overline{\boldsymbol{J}}_{K_1,k}$ and $\underline{\boldsymbol{J}}_{K_1,k}$ are the matrices formed from the first and the last $K_1 - k$ rows of the $K_1 \times K_1$ identity matrix, respectively. The Vandermonde structure of $\boldsymbol{A}_1$ implies that

$$\overline{\boldsymbol{A}}_{1,k} \boldsymbol{Q}_a^k = \underline{\boldsymbol{A}}_{1,k}, \ \text{for} \ \ k = 1, \ldots, K - 1 \tag{13}$$

where $\boldsymbol{Q}_a \triangleq \mathrm{diag}\{a_1, a_2, \ldots, a_P\}$. Let us define

$$\overline{\boldsymbol{H}}_{a,k} \triangleq \boldsymbol{B}_1 \circ \overline{\boldsymbol{A}}_{1,k} = (\boldsymbol{I}_{L_1} \circ \overline{\boldsymbol{J}}_{K_1,k}) \boldsymbol{H}_1 = \overline{\boldsymbol{T}}_{a,k} \boldsymbol{H}_1 \tag{14}$$
$$\underline{\boldsymbol{H}}_{a,k} \triangleq \boldsymbol{B}_1 \circ \underline{\boldsymbol{A}}_{1,k} = (\boldsymbol{I}_{L_1} \circ \underline{\boldsymbol{J}}_{K_1,k}) \boldsymbol{H}_1 = \underline{\boldsymbol{T}}_{a,k} \boldsymbol{H}_1 \tag{15}$$

where $\boldsymbol{I}_{L_1}$ denotes the $L_1 \times L_1$ identity matrix, $\overline{\boldsymbol{T}}_{a,k} \triangleq \boldsymbol{I}_{L_1} \circ \overline{\boldsymbol{J}}_{K_1,k}$, and $\underline{\boldsymbol{T}}_{a,k} \triangleq \boldsymbol{I}_{L_1} \circ \underline{\boldsymbol{J}}_{K_1,k}$. Similar to (14) and (15), we also define the $(K_1 - k) \times P$ matrices

$$\overline{\boldsymbol{E}}_{a,k} \triangleq \overline{\boldsymbol{T}}_{a,k} \boldsymbol{U}_1, \qquad \underline{\boldsymbol{E}}_{a,k} \triangleq \underline{\boldsymbol{T}}_{a,k} \boldsymbol{U}_1. \tag{16}$$

As $\boldsymbol{U}_1$ and $\boldsymbol{H}_1$ span the same signal subspace, there exists a full rank matrix $\boldsymbol{K}$ such that $\boldsymbol{H} = \boldsymbol{U}_1 \boldsymbol{K}$, and consequently, $\overline{\boldsymbol{H}}_{a,k} = \overline{\boldsymbol{E}}_{a,k} \boldsymbol{K}$ and $\underline{\boldsymbol{H}}_{a,k} = \underline{\boldsymbol{E}}_{a,k} \boldsymbol{K}$, for $k = 1, \ldots, K_1 - 1$. It follows from (13) that $\overline{\boldsymbol{H}}_{a,k} \boldsymbol{Q}_a^k = \underline{\boldsymbol{H}}_{a,k}$, and, consequently, that

$$\overline{\boldsymbol{E}}_{a,k} \boldsymbol{K} \boldsymbol{Q}_a^k = \underline{\boldsymbol{E}}_{a,k} \boldsymbol{K}, \quad k = 1, \ldots, K_1 - 1. \tag{17}$$

Equation (17) shows the MI structure of the signal subspace. With respect to the b-axis, the MI equation is expressed as

$$\overline{\boldsymbol{E}}_{b,l} \boldsymbol{K} \boldsymbol{Q}_b^l = \underline{\boldsymbol{E}}_{b,l} \boldsymbol{K}, \ \ \text{for} \ \ \ l = 1, \ldots, L_1 - 1 \tag{18}$$

where

$$\overline{\boldsymbol{E}}_{b,l} \triangleq \overline{\boldsymbol{T}}_{b,l} \boldsymbol{U}_1, \quad \underline{\boldsymbol{E}}_{b,l} \triangleq \underline{\boldsymbol{T}}_{b,l} \boldsymbol{U}_1, \quad \overline{\boldsymbol{T}}_{b,l} \triangleq \overline{\boldsymbol{J}}_{L_1,l} \circ \boldsymbol{I}_{K_1} \tag{19}$$

and

$$\boldsymbol{Q}_b \triangleq \mathrm{diag}\{b_1, b_2, \ldots, b_P\}. \tag{20}$$

Conventional harmonic retrieval techniques estimate the parameters from (17) and (18) using generalized eigenvalue decomposition (GEVD) (see [1] for the undamped sinusoid and [4] for damped sinusoid case) or joint diagonalization techniques [2]. The main advantage of using the joint diagonalization method or the GEVD-based approach is that in these techniques the MD parameter estimates are obtained jointly and the parameter association problem does not exist. A major drawback of these approaches, however, is that the computational cost associated with the use of the joint diagonalization or simultaneous GEVD algorithms is considerably high (except for the special case of simultaneous GEVD of two matrices [1, 4]). For joint GEVD of more than two matrices, iterative estimation schemes need to be applied [2], and global convergence of such schemes can not always be guaranteed, especially in the case of close eigenvalues. Moreover, joint diagonalization approaches rely only on the fact that the MI equations in (17) and (18) share a common diagonalization matrix $\boldsymbol{K}$ and ignore the relation that exists between $\boldsymbol{Q}_a^k$ (and $\boldsymbol{Q}_b^l$) for different values of k (and l).

4. ROOTING-BASED APPROACH

To overcome these difficulties in the damped harmonic case, we incorporate polynomial rooting instead of joint diagonalization to solve the MI equations. The generalized eigenvalue equation in (17) implies that the $P \times P$ matrices

$$\underline{\boldsymbol{E}}_{a,k}^H \underline{\boldsymbol{E}}_{a,k} - \underline{\boldsymbol{E}}_{a,k}^H \overline{\boldsymbol{E}}_{a,k} a^k, \quad \text{for} \quad k = 1, \ldots, K_1 - 1 \tag{21}$$

drop rank if $a \in \{a_1, \ldots, a_P\}$. A sum of the matrices (21) over different k yields a matrix polynomial of dimension $P \times P$ and degree $K_1 - 1$ given by

$$\boldsymbol{M}_a(a) = \sum_{k=1}^{K_1-1} \left(\underline{\boldsymbol{E}}_{a,k}^H \underline{\boldsymbol{E}}_{a,k} - \underline{\boldsymbol{E}}_{a,k}^H \overline{\boldsymbol{E}}_{a,k} a^k \right) \tag{22}$$

for which the following proposition holds.

Proposition P1: *If the column-reduced signal matrix $\underline{\boldsymbol{H}}_{a,1}$ in (15) is full column rank then the matrix polynomial $\boldsymbol{M}_a(a)$ is singular if a is a true generator, i.e. if $a \in \{a_1, \ldots, a_P\}$ and non-singular for any other values of a inside or on the unit circle.*

Proof: In order to proof **P1**, let us multiply $\boldsymbol{M}_a(a)$ from the left and the right with $\boldsymbol{K}^H$ and $\boldsymbol{K}$, respectively, to obtain

$$
\begin{aligned}
&\boldsymbol{K}^H \boldsymbol{M}_a(a)\boldsymbol{K} \\
&\quad= \sum_{k=1}^{K_1-1}\left(\boldsymbol{K}^H \underline{\boldsymbol{E}}_{a,k}^H \underline{\boldsymbol{E}}_{a,k}\boldsymbol{K} - \boldsymbol{K}^H \underline{\boldsymbol{E}}_{a,k}^H \overline{\boldsymbol{E}}_{a,k}\boldsymbol{K} a^k\right) \\
&\quad= \sum_{k=1}^{K_1-1}\left(\underline{\boldsymbol{H}}_{a,k}^H \underline{\boldsymbol{H}}_{a,k} - \underline{\boldsymbol{H}}_{a,k}^H \overline{\boldsymbol{H}}_{a,k} a^k\right) \\
&\quad= \sum_{k=1}^{K_1-1}\left(\underline{\boldsymbol{H}}_{a,k}^H \underline{\boldsymbol{H}}_{a,k}\left(\boldsymbol{I} - \boldsymbol{Q}_a^{-k} a^k\right)\right) \\
&\quad= \underbrace{\left[\sum_{k=1}^{K_1-1}\left(\underline{\boldsymbol{H}}_{a,k}^H \underline{\boldsymbol{H}}_{a,k}\sum_{m=1}^{k-1}\boldsymbol{Q}_a^{-m} a^m\right)\right]}_{\triangleq \boldsymbol{W}(a)}\left(\boldsymbol{I} - \boldsymbol{Q}_a^{-1} a\right).
\end{aligned}
$$

Since $\left(\boldsymbol{I} - \boldsymbol{Q}_a^{-1} a\right)$ becomes singular only at the true generators, it is sufficient to show that the residual matrix polynomial $\boldsymbol{W}(a)$ is non-singular inside or on the unit circle. In what follows, we will proof that for all $|a| \leq 1$ and for all nonzero $\boldsymbol{g} \in \mathbb{C}^{P\times 1}$ we have that $\mathrm{Re}\{\boldsymbol{g}^H \boldsymbol{W}(a)\boldsymbol{g}\} = \boldsymbol{g}^H \boldsymbol{W}_h(a)\boldsymbol{g} > 0$. This is equivalent to showing that the Hermitian part of $\boldsymbol{W}(a)$ given by

$$
\begin{aligned}
\boldsymbol{W}_h(a) \;&=\; \frac{1}{2}\sum_{k=1}^{K_1-1}\sum_{m=1}^{k-1}\underline{\boldsymbol{H}}_{a,k}^H \underline{\boldsymbol{H}}_{a,k}\boldsymbol{Q}_a^{-m} a^m \\
&+\; \frac{1}{2}\sum_{k=1}^{K_1-1}\sum_{m=1}^{k-1}\boldsymbol{Q}_a^{*-m} a^{*m}\underline{\boldsymbol{H}}_{a,k}^H \underline{\boldsymbol{H}}_{a,k}
\end{aligned}
\qquad (23)
$$

is positive definite. It is clear that the positive definiteness of $\boldsymbol{W}_h(a)$ inside and on the unit circle implies that $\boldsymbol{W}(a)$ is non-singular.

To prove that $\boldsymbol{W}_h(a)$ is positive definite for $|a| \leq 1$, one can show that

$$
\begin{aligned}
&2\boldsymbol{W}_h(a) \\
&= \sum_{k=1}^{K_1-2}\sum_{l=0}^{k-1}\sum_{n=0}^{k-1}(\boldsymbol{Q}_a^{*-1} a^*)^l \boldsymbol{Q}_a^{*\,K_1-1}\boldsymbol{B}_1^H \boldsymbol{B}_1 \boldsymbol{Q}_a^{K_1-1}(\boldsymbol{Q}_a^{-1} a)^n \\
&+ \sum_{l=0}^{K_1-2}\sum_{n=0}^{K_1-2}(\boldsymbol{Q}_a^{*-1} a^*)^l \boldsymbol{Q}_a^{*\,K_1-1}\boldsymbol{B}_1^H \boldsymbol{B}_1 \boldsymbol{Q}_a^{K_1-1}(\boldsymbol{Q}_a^{-1} a)^n \\
&+ \sum_{k=1}^{K_1-2}\sum_{m=k}^{K_1-2}\sum_{l=0}^{k-1}\sum_{n=0}^{k-1}(1-|a|^2)(\boldsymbol{Q}_a^{*-1} a^*)^l \boldsymbol{Q}_a^{*\,m}\boldsymbol{B}_1^H \boldsymbol{B}_1 \boldsymbol{Q}_a^{m}(\boldsymbol{Q}_a^{-1} a)^n \\
&+ \sum_{k=1}^{K_1-1}\sum_{m=k}^{K_1-1}\boldsymbol{Q}_a^{*\,m}\boldsymbol{B}_1^H \boldsymbol{B}_1 \boldsymbol{Q}_a^{m}.
\end{aligned}
$$

Since $1 - |a|^2 \geq 0$ for $|a| \leq 1$, we have that $\boldsymbol{W}_h(a)$ is positive definite inside and on the unit circle if

$$
\sum_{m=1}^{K_1-1}\boldsymbol{Q}_a^{*\,m}\boldsymbol{B}_1^H \boldsymbol{B}_1 \boldsymbol{Q}_a^{m} = \underline{\boldsymbol{H}}_{a,1}^H \underline{\boldsymbol{H}}_{a,1} > 0. \qquad (24)
$$

As $\underline{\boldsymbol{H}}_{a,1}$ is assumed to be full column rank, (24) always holds true. Therefore, $\boldsymbol{W}_h(a)$ for $|a| \leq 1$, and this completes the proof. ■

Similarly, with respect to the b-axis we have the following Proposition.

Proposition P2: *If the column-reduced signal matrix $\underline{\boldsymbol{H}}_{b,1}$ is full column rank, then the matrix polynomial*

$$
\boldsymbol{M}_b(b) = \sum_{l=1}^{L_1-1}\left(\underline{\boldsymbol{E}}_{b,k}^H \underline{\boldsymbol{E}}_{b,k} - \underline{\boldsymbol{E}}_{b,k}^H \overline{\boldsymbol{E}}_{b,k} b^k\right) \qquad (25)
$$

is singular if b is a true generator, i.e. if $b \in \{b_1, \ldots, b_P\}$ and non-singular for any other values of b inside or on the unit circle.

Using Proposition P1 (P2), one can estimate the set $\{a_p\}_{p=1}^P$ ($\{b_p\}_{p=1}^P$) by finding the values of a (b) inside the unit circle for which $\boldsymbol{M}_a(a)$ ($\boldsymbol{M}_b(b)$) is singular.

5. PARAMETER ASSOCIATION

In this section, we discuss the parameter association to find the correct pairs of estimates $\{(a_p, b_p)\}_{p=1}^P$ from the solutions that were separately obtained from rooting $\boldsymbol{M}_a(a)$ and $\boldsymbol{M}_b(b)$, respectively. From [7] we know that the P smallest (amplitude-wise) roots of the matrix polynomial $\boldsymbol{M}_a(a) = \sum_{k=0}^{K_1-1}\boldsymbol{F}_k a^k$ are given by the P minor eigenvalues of the associated companion matrix

$$
\mathcal{S}\{\boldsymbol{M}_a(a)\} \triangleq \qquad (26)
$$
$$
\begin{bmatrix}
\boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{0} & \cdots & \cdots & \cdots & \boldsymbol{I} \\
-\boldsymbol{F}_{K_1-1}^\dagger \boldsymbol{F}_0 & -\boldsymbol{F}_{K_1-1}^\dagger \boldsymbol{F}_1 & \cdots & \cdots & -\boldsymbol{F}_{K_1-1}^\dagger \boldsymbol{F}_{K_1-2}
\end{bmatrix}
$$

where † denotes the matrix pseudoinverse. It can readily be verified that the partitioned matrices

$$
\begin{aligned}
\boldsymbol{V}_a \;&=\; \left[\boldsymbol{V}_{a,0}^T, \boldsymbol{V}_{a,1}^T, \ldots, \boldsymbol{V}_{a,K_1-2}^T\right]^T \\
&=\; (K_1-1)^{-1}\left[\boldsymbol{K}^T, \boldsymbol{Q}_a \boldsymbol{K}^T, \ldots, \boldsymbol{Q}_a^{K_1-2}\boldsymbol{K}^T\right]^T \qquad (27) \\
\boldsymbol{V}_b \;&=\; \left[\boldsymbol{V}_{b,0}^T, \boldsymbol{V}_{b,1}^T, \ldots, \boldsymbol{V}_{b,L_1-2}^T\right]^T \\
&=\; (L_1-1)^{-1}\left[\boldsymbol{K}^T, \boldsymbol{Q}_b \boldsymbol{K}^T, \ldots, \boldsymbol{Q}_b^{L_1-2}\boldsymbol{K}^T\right]^T \qquad (28)
\end{aligned}
$$

represent the P minor eigenvectors of the companion matrices $\mathcal{S}\{\boldsymbol{M}_a(a)\}$ and $\mathcal{S}\{\boldsymbol{M}_b(b)\}$, respectively. Recall that the matrix $\boldsymbol{K}$ in (27) and (28) relates the signal matrix $\boldsymbol{H}_1$ with the singular vectors $\boldsymbol{U}_1$ as $\boldsymbol{H}_1 = \boldsymbol{U}_1\boldsymbol{K}$. Furthermore, $\boldsymbol{Q}_a$ and $\boldsymbol{Q}_b$ have the P eigenvalues of $\mathcal{S}\{\boldsymbol{M}_a(a)\}$ and $\mathcal{S}\{\boldsymbol{M}_b(b)\}$ (i.e., the generator estimates), respectively, on their main diagonals. Let us define $\boldsymbol{K}_a$ and $\boldsymbol{K}_b$ as

$$
\boldsymbol{K}_a \triangleq \sum_{k=0}^{K_1-2}\boldsymbol{V}_{a,k}\boldsymbol{Q}_a^{-k}, \qquad \boldsymbol{K}_b \triangleq \sum_{l=0}^{L_1-2}\boldsymbol{V}_{b,l}\boldsymbol{Q}_b^{-l}.
$$

If the diagonal elements of $\boldsymbol{Q}_a$ and $\boldsymbol{Q}_b$ have the correct association, then $\boldsymbol{K}_a = \boldsymbol{K}_b = \boldsymbol{K}$, otherwise $\boldsymbol{K}_a$ and $\boldsymbol{K}_b$ are column-wise permutations of each other. To find the correct parameter association, we compute $\boldsymbol{K}_a^H\boldsymbol{K}_b$ and find the element with the maximum magnitude in each particular row of $\boldsymbol{K}_a^H\boldsymbol{K}_b$. The row and column indices of this element show, respectively, which columns of $\boldsymbol{K}_a$ and $\boldsymbol{K}_b$ should be paired. Once the correct permutation between the columns $\boldsymbol{K}_a$ and $\boldsymbol{K}_b$ is found, one can pair the diagonal entries of $\boldsymbol{Q}_a$ and $\boldsymbol{Q}_b$, because $\boldsymbol{Q}_a$ and $\boldsymbol{Q}_b$ are associated to each other according to the column permutation of $\boldsymbol{K}_a$ and $\boldsymbol{K}_b$.

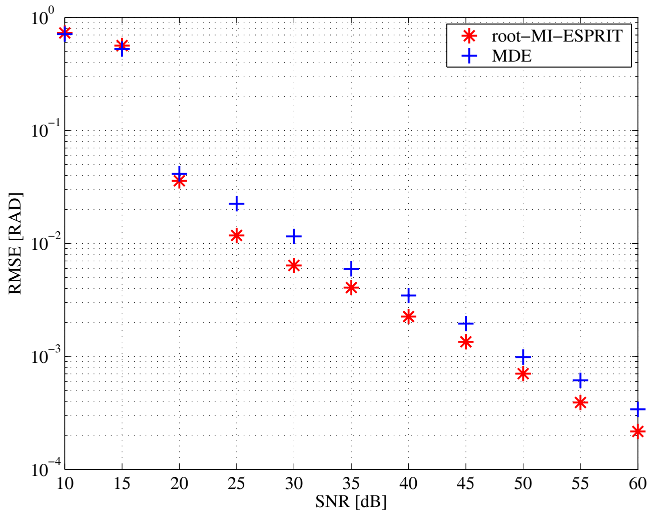

Fig. 1. RMSE of $\hat{\omega}_1$ versus SNR.

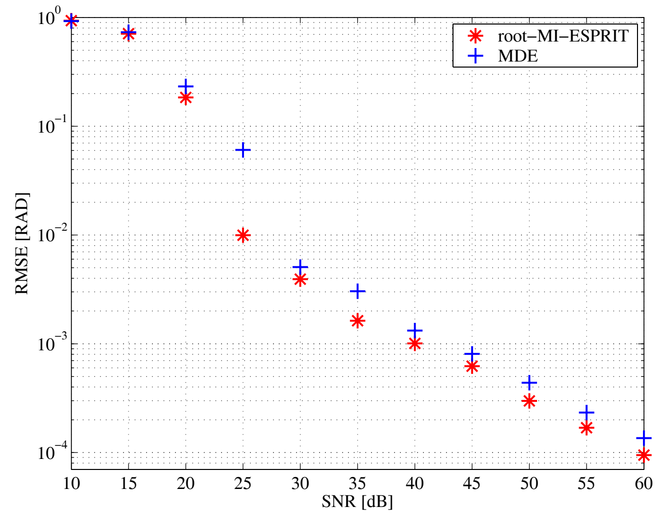

Fig. 2. RMSE of $\hat{\omega}_2$ versus SNR.

6. SIMULATION RESULTS AND DISCUSSION

We compare the estimation performance of the proposed root-MI-ESPRIT algorithm and the MDE technique of [4]. The latter technique is known to be one of the best subspace algorithms developed for harmonic retrieval of damped exponentials. We assume five equi-power 2D damped harmonics whose parameters given in the table below. These harmonics are sampled over a rectangular grid with support of $K = L = 24$. The root-mean-square errors (RMSEs) of the frequency parameters ω_1 and ω_2 are plotted versus the SNR in Figs. 1 and 2, respectively. All results are averaged over 100 independent simulation runs.

p	$\alpha_{1,p}$	$\omega_{1,p}$	$\alpha_{2,p}$	$\omega_{2,p}$
1	-0.2	0.1π	-0.1	0.1π
2	-0.0	0.3π	-0.0	0.1π
3	-0.05	0.2π	-0.02	0.25π
4	-0.02	0.05π	-0.02	0.3π
5	-0.01	0.06π	-0.02	0.31π

From these figures, we observe that root-MI-ESPRIT uniformly outperforms MDE at a comparable computational complexity. These performance improvements can be explained by the fact that in root-MI-ESPRIT not only a single invariance but the full MI structure of the data is exploited.

7. CONCLUSIONS

A novel 2D harmonic retrieval algorithm has been proposed that exploits the entire MI structure of the data model. Unlike previous methods that rely on joint eigendecomposition of multiple ESPRIT matrices, a different approach is taken here. Instead of searching for common eigenvectors that simultaneously solve the invariance equations, the relation between the eigenvalues corresponding to different invariance equations is exploited. The solutions along each harmonic axis are separately obtained from the roots of a matrix polynomial and, in a post-processing step, are associated (paired) according to their eigenvectors. Our technique for solving the MI equations is computationally attractive and establishes a link between ESPRIT-type algorithms and rooting-based algorithms for damped and undamped harmonic retrieval. The generalization of the proposed algorithm to the case of higher dimensions is straightforward.

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