MULTICHANNEL FIR EXACT DECONVOLUTION IN MULTIPLE VARIABLES

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ABSTRACT

We present a general framework for multichannel exact deconvolution with multivariate finite impulse response (FIR) convolution and deconvolution filters using algebraic geometry. Previous work formulates the problem of multichannel FIR deconvolution into that of the left inverse of a convolution matrix which is solved by linear algebra. However, this approach requires the prior information of the support of deconvolution filters. Using algebraic geometry, we find a necessary and sufficient existence condition of FIR deconvolution filters and propose a simple algorithm based on the Gröbner basis to compute deconvolution filters. This computation algorithm obtains deconvolution filters with either minimal order or minimum number of nonzero coefficients, and no prior information of the support is required. Simulation results show that due to the smaller size of deconvolution filters our approach achieves better results than the liner algebra approach under the impulsive noise environment.

1. INTRODUCTION

Over the last decade, the theory and applications of multichannel deconvolution have grown rapidly, such as channel equalization for multiple antennas [1], multichannel image deconvolution [2], and polarimetric calibration of radars [3]. In these applications, the original data is filtered by multiple finite impulse response (FIR) filters with possible additive noise, and the goal of the multichannel deconvolution is to reconstruct the original data given the multiple filtered data as shown in Fig. 1.

Harikumar and Bresler considered the multichannel two-variate FIR exact deconvolution problem where the deconvolution filters are FIR and the reconstruction data equals to the original data when there is no additive noise [2]. Such FIR exact deconvolution is more computationally efficient than traditional least-square solutions. Harikumar and Bresler proposed a linear algebra algorithm to compute the deconvolution filters, which requires the prior information on the support size of deconvolution filters. In general, the resulting filters do not have minimum support. Rajagopal and Potter applied the Gröbner basis to compute equalizers without the prior knowledge of the support of deconvolution filters [3]. The filters they considered are polynomial or causal filters, while the filters we consider here are general FIR filters. Such filters have been used in many deconvolution applications, for example, image deconvolution.

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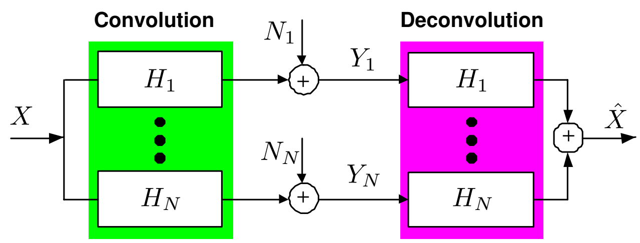

Fig. 1. An N-channel deconvolution. Original data X is filtered by N convolution filters $\{H_1, \ldots, H_N\}$ with possible additive noise. Reconstruction data $\hat{X}$ is the linear combination of the deconvolution by N deconvolution filters $\{G_1, \ldots, G_N\}$ from N outputs $\{Y_1, \ldots, Y_N\}$.

In this work, we address the multichannel multivariate FIR exact deconvolution problem using algebraic geometry [4]. We derive a sufficient and necessary condition for the existence of FIR deconvolution filters and propose a simple algorithm based on the Gröbner basis to compute the deconvolution filters without the prior knowledge of the support of the filters.

The rest of the paper is organized as follows. In Section 2, we briefly introduce the system model and the Gröbner basis theory. We propose the existence condition of deconvolution filters in Section 3 and a computation algorithm based on the Gröbner basis in Section 4. The simulation results under the impulsive noise environment are given in Section 5. Due to the space limitation, we only state the results here and refer readers to [5] for proofs.

2. PRELIMINARIES

We start with notations. Throughout the paper, we will always refer to M as the number of variables, and N as the number of channels. We denote sets, vectors, or matrices by boldface letters, for example, $\boldsymbol{z}$ stands for an M-dimensional complex variable $\boldsymbol{z} = [z_1, \ldots, z_M]^T$, and raising $\boldsymbol{z}$ to an M-dimensional integer vector $\boldsymbol{k} = [k_1, \ldots, k_M]^T$ yields $\boldsymbol{z}^{\boldsymbol{k}} = \prod_{i=1}^{M} z_i^{k_i}$. We denote the z-transform of signals or filters by uppercase letters, and occasionally we will suppress the variable $\boldsymbol{z}$ for simplicity.

Exact deconvolution requires that the reconstruction data $\hat{X}$ equals to the original data X in Fig. 1 when there are no noises. In the z-domain, the exact deconvolution condition is equivalent to

$$\sum_{i=1}^{N} H_i(\boldsymbol{z}) G_i(\boldsymbol{z}) = 1. \tag{1}$$

In the multichannel exact deconvolution problem, the convolution filters $\{H_i\}$ are given, and the goal is to find the deconvolution filters $\{G_i\}$ satisfying (1). For the univariate polynomial case, (1) has a solution if the common greatest divisor of $\{H_1, \ldots, H_N\}$ is 1 and we can use the univariate division algorithm to find a solution [6, 7]. Unfortunately, these algorithms fail for multivariate polynomials. One important reason is that the remainder of the multivariate division algorithm depends on the order of H_i, while the remainder in the univariate division algorithm is uniquely determined.

Gröbner basis extends the univariate polynomial theory and algorithms such as the common greatest divisor and the division algorithm to multivariate polynomials [8, 9]. A Gröbner basis is a set of polynomials such that given any polynomial its remainder related to this set is uniquely determined, regardless of the order of the polynomials. Moreover, given a polynomial set $\{H_1, \ldots, H_N\}$, there exists a unique reduced Gröbner basis $\{B_1, \ldots, B_n\}$ and an $n \times N$ (polynomial) transform matrix $\boldsymbol{W}$ such that,

$$B_i = \sum_{j=1}^{N} W_{i,j} H_j, \quad \text{for } 1 \leq i \leq n, \tag{2}$$

where $W_{i,j}$ is the element of $\boldsymbol{W}$ at (i, j). Note that the size of the reduced Gröbner basis may differ from the size of the given polynomial set. The reduced Gröbner basis and the associated transform matrix can be computed by the Buchberger's algorithm, which has been implemented by many computer algebra systems such as *Macaulay2*.

3. INVERTIBLE CONDITIONS

In the multichannel FIR exact deconvolution problem, the convolution filters $\{H_1, \ldots, H_N\}$ are given FIR filters, and our task is to find a set of FIR deconvolution filters $\{G_1, \ldots, G_N\}$ satisfying (1). However, the set of FIR deconvolution filters does not always exist.

Definition 1 *A set of FIR filters $\{H_1, \ldots, H_N\}$ is said to be* FIR invertible *if there exists a set of FIR filters $\{G_1, \ldots, G_N\}$ satisfying the perfect reconstruction condition (1).*

Algebraic geometry and Gröbner basis are powerful tools for multivariate polynomials. To apply these tools, we need to convert the FIR problem into a polynomial problem. One key observation is that we can convert any FIR filter into a polynomial by multiplying it with a monomial with high enough degree (equivalent to shifting the origin so that all filters are causal). In this way, we convert both $\{H_i\}$ and $\{G_i\}$ into polynomials by multiplying both sides of (1) with a monomial. Therefore, the exact deconvolution condition for the FIR filters in (1) is equivalent to a condition for the polynomial filters:

$$\sum_{i=1}^{N} H_i(\boldsymbol{z}) G_i(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{m}}, \quad \text{for some integer vector } \boldsymbol{m}. \tag{3}$$

Without loss of generality, we assume that the convolution filters $\{H_1, \ldots, H_N\}$ are polynomials in this section. Now we apply algebraic geometry to (3) and obtain an existence condition.

Theorem 1 *([5]) Suppose $\{H_1(\boldsymbol{z}), H_2(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z})\}$ is a set of multivariate polynomials. Then it is FIR invertible* if and only if *every complex solution of the system of equations*

$$\{H_1(\boldsymbol{z}) = 0, \ \ldots, \ H_N(\boldsymbol{z}) = 0\}$$

is weak-zero, that is, at least one of its element is zero.

Theorem 1 gives a sufficient and necessary condition for the FIR exact deconvolution. However, this theorem does not provide a practical criterion since it is generally difficult to find all complex solutions for a system of polynomial equations. Using the Gröbner basis, we obtain a computational test criterion for the existence of FIR deconvolution filters.

Theorem 2 *([5]) Suppose $\{H_1(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z})\}$ is a set of multivariate polynomials. Then it is FIR invertible* if and only if *the reduced Gröbner basis of $\{H_1(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z}), 1 - z_1 \cdots z_{M+1}\}$ is $\{1\}$, where $\boldsymbol{z} = [z_1, \ldots, z_M]$ and z_{M+1} is a new variable.*

The key technique in this theorem is the introduction of a new variable z_{M+1} that maps the FIR deconvolution into a polynomial one. To illustrate Theorem 1 and Theorem 2, we give an example.

Example 1 *Let $H_1(z_1, z_2) = z_1 + z_2^2 - 1$ and $H_2(z_1, z_2) = z_1 + z_2 - 1$. They have two common zeros, $\boldsymbol{z} = (1, 0)$ and $\boldsymbol{z} = (0, 1)$, but both of them are weak-zero. By Theorem 1, the set $\{H_1, H_2\}$ is FIR invertible. The reduced Gröbner basis of $\{H_1, H_2, 1 - z_1 z_2 z_3\}$ is $\{1\}$. By Theorem 2, $\{H_1, H_2\}$ is FIR invertible.*

4. COMPUTING DECONVOLUTION FILTERS

In this section, we present an algorithm to test the invertibility and compute a set of FIR deconvolution filters. We also present the characterization of all sets of FIR deconvolution filters.

By Theorem 2, if the set of convolution filters is FIR invertible, the reduced Gröbner basis of $\{H_1, \ldots, H_N, 1 - z_1 \cdots z_{M+1}\}$ is $\{1\}$. Suppose the corresponding transform matrix is $(W_1, \ldots, W_{N+1})$. Then (2) becomes

$$\sum_{i=1}^{N} H_i(\boldsymbol{z}) W_i(\boldsymbol{z}, z_{M+1}) + (1 - \prod_{j=1}^{M+1} z_j) W_{N+1}(\boldsymbol{z}, z_{M+1}) = 1. \tag{4}$$

Set $z_{M+1} = z_1^{-1} \cdots z_M^{-1}$, and then (4) becomes

$$\sum_{i=1}^{N} H_i(\boldsymbol{z}) W_i(\boldsymbol{z}, \prod_{j=1}^{M} z_j^{-1}) = 1,$$

which implies $\{W_1(\boldsymbol{z}, z_1^{-1} \cdots z_M^{-1}), \ldots, W_N(\boldsymbol{z}, z_1^{-1} \cdots z_M^{-1})\}$ is a set of FIR deconvolution filters.

Algorithm 1 *([5]) The test and computational algorithm for a set of FIR deconvolution filters is given as follows.*
Input: a set of FIR convolution filters $\{H_1, \ldots, H_N\}$.
Output: a set of FIR deconvolution filters, if it exists.

1. *Multiply $\{H_i\}$ by a common monomial $\boldsymbol{z}^{\boldsymbol{m}_0}$ such that $\{H_i\}$ are polynomials.*
2. *Use the Buchberger's algorithm to compute the reduced Gröbner basis of $\{H_1, \ldots, H_N, 1 - z_1 \cdots z_{M+1}\}$ and the associated transform matrix $\boldsymbol{W}$.*
3. *If the reduced Gröbner basis is $\{1\}$, Simplify*

   $$\boldsymbol{G} = \{W_1(\boldsymbol{z}, z_1^{-1} \cdots z_M^{-1}), \ldots, W_N(\boldsymbol{z}, z_1^{-1} \cdots z_M^{-1})\}$$

   and output $\boldsymbol{z}^{-\boldsymbol{m}_0} \boldsymbol{G}$.

   Otherwise, there is no solution.

Algorithm 1 generates a set of FIR deconvolution filters without the prior knowledge of the size of the deconvolution filters. Moreover, by changing the order of polynomials in $\{H_1, \ldots, H_N, 1 - z_1 \cdots z_{M+1}\}$, the deconvolution filters can be computed with either minimal order or minimum number of coefficients. To illustrate this algorithm, we give an example.

Example 2 *Consider the polynomial set $\{H_1, H_2\}$ in Example 1. By the Buchberger's algorithm, the reduced Gröbner basis of $\{H_1, H_2, 1 - z_1 z_2 z_3\}$ is $\{1\}$ and the transform matrix is $(-z_3, z_3 + z_2 z_3, 1)$. By Algorithm 1, we obtain a set of deconvolution filters $\{-z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-1} + z_1^{-1}\}$. It can be verified that this set has minimum number of coefficients.*

The set of FIR deconvolution filters for a given set of FIR convolution filters is not unique. In the following, we characterize all sets of FIR deconvolution filters using one set of FIR deconvolution filters.

Using the matrix format, we express (3) as

$$\mathbf{H}^T(\boldsymbol{z}) \mathbf{G}(\boldsymbol{z}) = 1, \tag{5}$$

where $\mathbf{H}(\boldsymbol{z})$ and $\mathbf{G}(\boldsymbol{z})$ are $N \times 1$ vectors of FIR convolution filters and deconvolution filters respectively.

Theorem 3 *([5]) Suppose $\mathbf{H}(\boldsymbol{z})$ is a given vector of FIR convolution filters and $\mathbf{G}_p(\boldsymbol{z})$ is a vector of FIR deconvolution filters satisfying (5). Then a vector of FIR deconvolution filters $\mathbf{G}(\boldsymbol{z})$ also satisfies (5)* if and only if *$\mathbf{G}(\boldsymbol{z})$ can be written as*

$$\mathbf{G}(\boldsymbol{z}) = \mathbf{G}_p(\boldsymbol{z}) + \big(\mathbf{I} - \mathbf{G}_p(\boldsymbol{z}) \mathbf{H}^T(\boldsymbol{z})\big) \mathbf{S}(\boldsymbol{z}),$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{S}(\boldsymbol{z})$ is an $N \times 1$ FIR vector.

5. SIMULATIONS

We illustrate the simulation results under the impulsive noise environment, which is common in many deconvolution applications such as equalization. The signal-to-noise ratio (SNR) is defined as

$$SNR = 10 \log_{10}\left(\frac{\sum_{i=1}^{p} \|Y_i\|^2}{\sigma^2}\right),$$

The impulsive noise N is defined as

$$N = \begin{cases} 0, & \text{with probability} \quad (1 - \alpha), \\ \sim \mathcal{N}(0, \sigma^2), & \text{with probability} \quad \alpha, \end{cases}$$

where α is the occurrence probability of the impulsive noise. In the simulations, we choose the SNR to be -30 dB and α to be 0.0001. The original image and the noisy convolution outputs are shown in Fig. 2(a) and Fig. 2(b)-(d).

The three convolution filters are given as

$$H_1 = \begin{pmatrix} 4 & 0 & -20 & 0 & 16 \\ 0 & 20 & 20 & -32 & -32 \\ -5 & -10 & 19 & 48 & 24 \\ 0 & -8 & -24 & -24 & -8 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

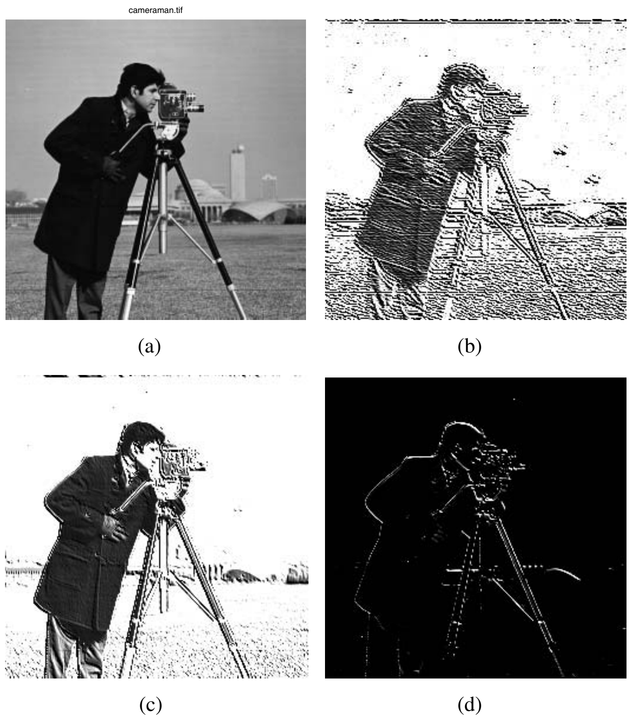

Fig. 2. (a) Original image of size 256×256. (b)-(d) Convolution outputs imposed by additive impulsive Gaussian noises (SNR=-30 dB, $\alpha = 0.0001$).

and

$$H_2 = \begin{pmatrix} 3 & -8 & 4 \\ 4 & 0 & -4 \\ 1 & 2 & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} -3 & -4 & 4 \\ 2 & -2 & -4 \\ 1 & 2 & 1 \end{pmatrix}.$$

We apply Algorithm 1 and compute a set of deconvolution filters, which are exact solutions given as

$$G_1 = \frac{1}{40}, \quad G_2 = \begin{pmatrix} \frac{3}{20} & \frac{1}{10} & \frac{3}{20} & \frac{1}{10} \\ -\frac{1}{20} & -\frac{1}{5} & -\frac{3}{10} & -\frac{3}{20} \\ \frac{3}{80} & \frac{3}{20} & \frac{3}{16} & \frac{3}{40} \\ -\frac{1}{80} & -\frac{3}{80} & -\frac{3}{80} & -\frac{1}{80} \end{pmatrix},$$

and

$$G_3 = \begin{pmatrix} -\frac{3}{20} & -\frac{1}{10} & -\frac{3}{20} & -\frac{1}{10} \\ \frac{1}{20} & \frac{1}{5} & \frac{3}{10} & \frac{3}{20} \\ -\frac{3}{80} & -\frac{3}{20} & -\frac{3}{16} & -\frac{3}{40} \\ \frac{1}{80} & \frac{3}{80} & \frac{3}{80} & \frac{1}{80} \end{pmatrix}.$$

The first deconvolution filter is just a scalar. The size of rest two filters is 4×4. For comparison, we compute the size of deconvolution filters using the estimates in [2], which is 4×8. Then we compute a set of deconvolution filters of size 4×8 by the linear algebra approach, which gives numerical solutions. Hence, Algorithm 1 obtains deconvolution filters with smaller size. Actually, it can be proved that the obtained filters have minimum number of coefficients.

Then we use two sets of deconvolution filters to reconstruct the original image. The reconstruction images by the linear algebra approach is shown in Fig. 3(a) and our proposed approach in Fig. 3(b). Since both sets of deconvolution filters are FIR, the impulsive noise have been isolated from propagation in both reconstruction images. However, the impulsive noises in Fig. 3(b) are smaller than those in Fig. 3(a). The reason is that the deconvolution filters obtained by our proposed approach have smaller supports than those obtained by the linear algebra approach.

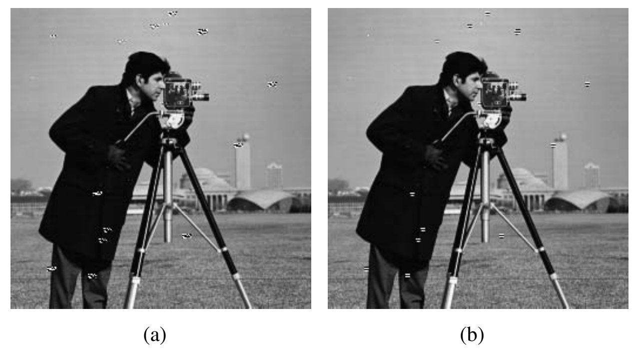

Fig. 3. Reconstruction images by the FIR deconvolution filters obtained by: (a) Linear algebra approach; (b) Our proposed approach.

6. REFERENCES

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