APPLICATION OF THE BOOTSTRAP TO SOURCE DETECTION IN NONUNIFORM NOISE

*Saïd Aouada, Danail Traskov, Nico d'Heureuse, and Abdelhak M. Zoubir**

Signal Processing Group, Darmstadt University of Technology,
Merckstrasse 25, D-64283, Darmstadt, Germany.
{aouada,zoubir}@ieee.org

ABSTRACT

We consider the problem of source number estimation in the presence of unknown spatially nonuniform noise. A sequential hypothesis test is formulated based on Gerschgorin's theorem. As no assumption is made on the distribution of the noise or the signals, the bootstrap is used to estimate the distribution of the proposed test statistics. Performance of the new detector is assessed through simulations and its advantage is illustrated in different scenarios against a Gerschgorin-based information criterion.

1. INTRODUCTION

Model-order selection techniques apply to array signal processing for the purpose of determining the number of signals impinging on a sensor array [1, 2]. These methods range from hypothesis testing [3]-[5] to information theoretic criteria [6, 7]. When the noise powers are different from one sensor to another, most classical detectors fail to correctly estimate the number of sources and few dedicated detectors are available.

A hypothesis test was proposed in [4] for the case of spatially correlated noise under the assumption that the noise covariance matrix has a band structure. This approach can be extended to nonuniform noise but can use half of the available sensors at most, thus making it a very restrictive approach. In [7], an information criterion was proposed based on Gerschgorin radii to tackle spatial nonuniformity. However, similarly to the classical approaches, the method fails as it incorporates the erroneously ordered eigenvalues of the data covariance matrix.

In [8], a more appropriate derivation was carried-out based on Gerschgorin's theorem, where the contribution of the unordered eigenvalues was suppressed. The resulting Log-Likelihood function demonstrates robustness to the variation of the powers of the noise. When combined to appropriately defined penalty functions, information theoretic criteria can be obtained for both the nonuniform and the special uniform noise cases. The resulting functions retain the asymptotic properties of the classical detectors, and versions for small samples were also suggested [9].

The main drawback of Gerschgorin-based information criteria is that the goodness-of-fit term is specifically derived for Gaussian signals. If the assumption of Gaussianity is not verified, the behavior of the aforementioned detectors is expected to degrade significantly, especially for relatively low Signal to Noise Ratios (SNR) and small sample sizes.

In what follows, we propose an alternative detection scheme based on a sequential hypothesis test where no assumption on the distribution of the data is made. The proposed test statistics are based on the discriminating property of the Gerschgorin radii [8] and their distribution under the null is estimated using the Bootstrap. Thus, the new method deals with deviations from both Gaussianity and asymptotic conditions.

*This work is partly supported by the DSO National Labs, Singapore.

2. DATA MODEL

Consider p narrowband signals impinging on an M-element array. The number of sources p is unknown and is to be estimated. It is assumed that $p < M$. The sources are assumed to be coplanar and located in the far field.

The received signal vector at instant t can be modeled as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \qquad t = 1, \ldots, L \tag{1}$$

where $\mathbf{A}$ is the $(M \times p)$-dimensional array steering matrix representing the array spatial response to the p wavefronts, $\mathbf{s}(t)$ is the p-dimensional vector of the source signals and $\mathbf{n}(t)$ is the M-dimensional vector of sensor noise.

The additive noise $\mathbf{n}(t)$ is assumed to be a spatially and temporally white process with an unknown diagonal covariance matrix $\mathbf{Q}$, i.e.,

$$\mathbf{Q} = \mathsf{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \text{diag}\left\{\mathbf{q}\right\} \tag{2}$$

where $(.)^H$ denotes Hermitian transpose and $\mathsf{E}(.)$ expectation. Spatial non-uniformity of the noise is modeled by different powers through the sensors, such that

$$\mathbf{q} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_M^2]^T. \tag{3}$$

The source signals and the noise are assumed to be uncorrelated and their respective distributions are unknown. Consequently, the array covariance matrix is given by

$$\mathbf{R} = \mathsf{E}\left\{\mathbf{x}(t)\mathbf{x}^H(t)\right\} = \mathbf{A}\mathbf{R}_\mathbf{s}\mathbf{A}^H + \mathbf{Q} \tag{4}$$

where $\mathbf{R}_\mathbf{s} = \mathsf{E}\left\{\mathbf{s}(t)\mathbf{s}^H(t)\right\}$ is the source signal covariance matrix.

3. SOURCE NUMBER DETECTION

3.1. Covariance Matrix Transformation

As the noise powers are different from one sensor to another, the effect of noise due to the individual sensors is alleviated by neutralizing one arbitrary element from the array [7, 8]. If, say, the

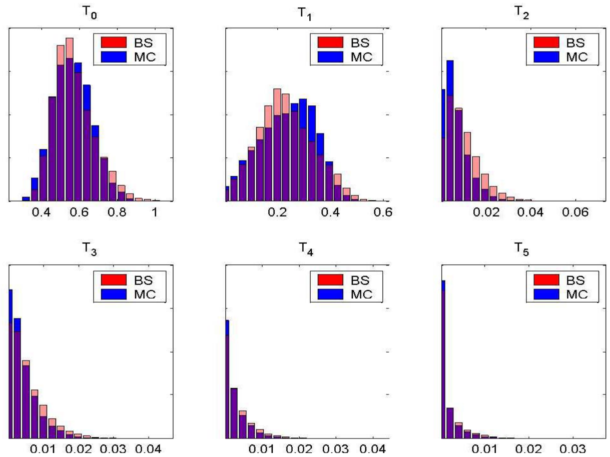


Fig. 1. Histogram of the test statistics using the Bootstrap (Empirical probability vs value of the test statistics).

contribution of the k-th array element is suppressed, a unitary transformation matrix $\mathbf{U}$ can be defined such that, when applied to the original covariance matrix $\mathbf{R}$, the following matrix is obtained [7, 8]

$$
\begin{aligned}
\boldsymbol{\mathcal{R}} &= \mathbf{U}^H \mathbf{R} \mathbf{U} \\
&= \begin{bmatrix}
\lambda_1 & \dots & 0 & c_1^* & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \lambda_{k-1} & c_{k-1}^* & 0 & \dots & 0 \\
c_1 & \dots & c_{k-1} & r_{(k,k)} & c_k & \dots & c_{M-1} \\
0 & \dots & 0 & c_k^* & \lambda_k & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & c_{M-1}^* & 0 & \dots & \lambda_{M-1}
\end{bmatrix}
\end{aligned} \tag{5}
$$

where $(.)^*$ stands for complex conjugate, λ_m,, $m = 1, \dots, M-1$, are the eigenvalues of the $(M-1)$-dimensional reduced covariance matrix $\mathbf{R}_k$, and $r_{(k,k)}$ is the element of $\mathbf{R}$ of index (k, k), corresponding to the k-th suppressed array element.

The elements c_m, $m = 1, \dots, M-1$, in (5) are the projection of the k-th column of $\mathbf{R}$ onto eigenvector $\mathbf{e}_m$, corresponding to eigenvalue λ_m. Their magnitudes $\rho_m = |c_m|$ are the Gerschgorin radii of matrix $\boldsymbol{\mathcal{R}}$ [7, 8].

Based on the information contained in the elements ρ_m, $m = 1, \dots, M-1$, it is possible to separate the noise and signal subspaces after ordering the elements as [8]

$$\rho_1 \geq \rho_2 \geq \dots \geq \rho_p \geq \rho_{p+1} = \rho_{p+2} = \dots = \rho_{M-1} = 0 \tag{6}$$

where the first p elements, $\rho_1, \dots, \rho_p$, correspond to the signal subspace.

3.2. Sequential Hypothesis Test

In practice, due to the finite data length, the sample covariance matrix $\hat{\mathbf{R}}$ is used, leading to radii $\hat{\rho}_m$, $m = 1, \dots, M-1$.

From Equation (6), it can be deduced that estimating the number of sources p can be achieved by checking simultaneously for zero the Gerschgorin radii $\hat{\rho}_m$, $m = p+1, \dots, M-1$, corresponding to the noise-only subspace. This suggests the following test statistics:

$$
\begin{aligned}
T_{1_i} &= \sum_{m=i+1}^{M-1} \hat{\rho}_m^2 && (7) \\
T_{2_i} &= \left(\frac{1}{M-1-i} \sum_{m=i+1}^{M-1} \hat{\rho}_m \right) - \left(\prod_{m=i+1}^{M-1} \hat{\rho}_m^{\frac{1}{M-1-i}} \right) && (8) \\
& \text{for } \; i = 0, \dots, M-2.
\end{aligned}
$$

In practice, the values of both statistics T_{1_i} (cumulated squared radii) and T_{2_i} (difference between radii arithmetic and geometric means) will be small (close to zero) if all the Gerschgorin radii that they enclose correspond to the noise-only subspace, and significantly greater than zero otherwise. Formulating this variation in a sequence of independent hypothesis tests translates to

$$
\begin{array}{lcrcl}
\mathsf{H}_0 & : & \rho_1 = \dots = \rho_{M-1} & = & 0 \\
\vdots & & & \vdots & \\
\mathsf{H}_i & : & \rho_{i+1} = \dots = \rho_{M-1} & = & 0 \\
\vdots & & & \vdots & \\
\mathsf{H}_{M-2} & : & \rho_{M-1} & = & 0
\end{array} \tag{9}
$$

The test starts by checking that the global null, H_0, is verified, i.e., that no sources are present. If the hypothesis is not rejected, then the estimated number of sources is $\hat{p} = 0$. If H_0 is rejected, and its alternative, $\mathsf{K}_0 \,:\, (\hat{\rho}_1 \neq 0$ or $\dots$ or $\hat{\rho}_{M-1} \neq 0)$, is retained, then it is known that at least one source is present. However no information is directly deduced about the actual number of sources. Thus, by stepping through the hypotheses H_i, $i = 0, \dots, M-2$, the contribution of the largest Gerschgorin radius is eliminated sequentially from the test statistic and the dimension of the candidate noise-only subspace over which the null is tested, is reduced. The test stops when a hypothesis is accepted, or when it reaches H_{M-2}, indicating that there are $M-2$ sources.

For a given level of significance α, when no sources are present (global null), the probability of correctly deciding that $\hat{p} = 0$ must be maintained at $1 - \alpha$. Thus, each hypothesis in (9) is tested at a level α.

3.3. Bootstrap: Estimation of the Distribution

At this stage, we make no assumption about the distribution of the data and use the bootstrap to estimate the distribution under the null of the test statistics $T_{1,2_i}$ [2, 5]. The principle of the bootstrap is that the data sample represents an empirical estimate of the true distribution. Thus, resampling from this estimate creates bootstrap data sets which are used to conduct inference. The bootstrap therefore avoids errors due to the asymptotic approximations in small sample scenarios.

An example of estimated distributions corresponding to the statistics $T_{2,i}$, $i = 0, \dots, 5$, is shown in Figure 1, where $M-1 = 6$, $p = 2$, $L = 100$, for SNR=3 dB at a Worst Noise Power Ratio[1] (WNPR) of 10. In Figure 1, BS and MC stand for bootstrap and Monte-Carlo, respectively. Empirical tests show that the distributions under the null (noise-only subspace) can be accurately estimated using the bootstrap. For similar cases, [5] and the references therein highlight the issue of improving the estimates of

[1]WNPR is defined in [11] as WNPR $= \sigma^2_{\text{max}} / \sigma^2_{\text{min}}$.

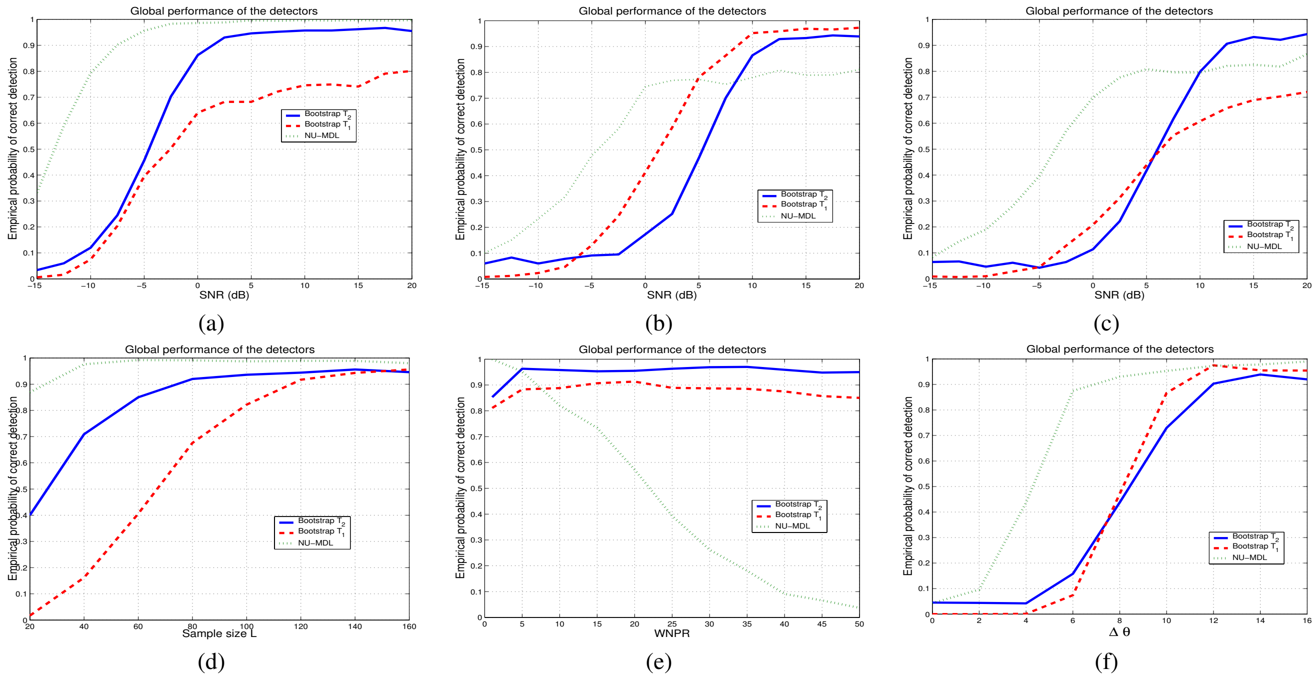


Fig. 2. Performance of the detectors vs: **(a)** SNR, **(b)** SNR (Gaussian signals in Laplacian noise), **(c)** SNR (Laplacian signals in Gaussian noise), **(d)** L, **(e)** WNPR, **(f)** $\Delta\theta$.

the empirical distributions of complicated test statistics (i.e., functions of the eigenvalues), using for example, fewer resamples or subsampling.

Let the b-th bootstrap resample of the data be denoted $\mathbf{x}_b^\star(t)$, for $t = 1, \ldots, L$ and $b = 1, \ldots, B$. From B bootstrap resamples, the estimate of the empirical distribution of the test statistics under the null can be obtained as [5, 10]

$$\hat{T}_i^{\mathsf{H}}(b) = T_i^\star(b) - T_i; \qquad b = 1, \ldots, B \tag{10}$$

where T_i is the test statistics evaluated from the data $\mathbf{x}(t)$, while $T_i^\star(b)$ is the test statistics evaluated from the resample $\mathbf{x}_b^\star(t)$. The significance values for the hypothesis tests of (9) are given by [5, 10]

$$P_i = \frac{1}{B}\sum_{b=1}^{B} I\left(|T_i| \leq |\hat{T}_i^{\mathsf{H}}(b)|\right) \tag{11}$$

with $I(.)$ being the indicator function.

Thus, with a level of significance α, starting from $i = 0$, if $P_i \geq \alpha$ then H_i is accepted, otherwise set $i = i+1$ and repeat the test.

4. SIMULATION RESULTS

We show the global performance of the bootstrap-based detectors (T_1 and T_2) and compare it to the previously proposed Gerschgorin-based detector, namely the NU-MDL [8].

A Uniform Linear Array (ULA) is assumed with $M = 5$ sensors. The true number of sources is $p = 2$. The samples are bootstrapped $B = 100$ times and a global level of significance $\alpha = 2\%$ is chosen. All the examples illustrate the empirical probability of correct detection resulting from 1000 Monte-Carlo runs. Unless indicated otherwise, the noise and the signals are assumed to be Gaussian.

Figure 2 (a) illustrates the performance with respect to SNR. The fixed parameters are the number of snapshots $L = 100$, the angles of arrival $\boldsymbol{\theta} = [4°, 20°]^T$, and WNPR=10, whereas SNR is variable. With Gaussian data, as expected, the NU-MDL performs better than T_1 and T_2 as it employs an accurate goodness-of-fit term. The latter is based on the Gaussian distribution and thus correctly fits the data. In addition to that, the bootstrap detectors as hypothesis tests are different from the NU-MDL information criterion which is known for its consistency properties. The bootstrap detectors perform well without any *a priori* knowledge of the distribution of the data. Note the relative improvement in performance of T_2 as opposed to T_1. This can be explained by the fact that when testing for the null (checking for $T_{1,2}$ to be zero), the structure of T_2 (difference between two metrics) allows it to be closer to zero "more often" than T_1.

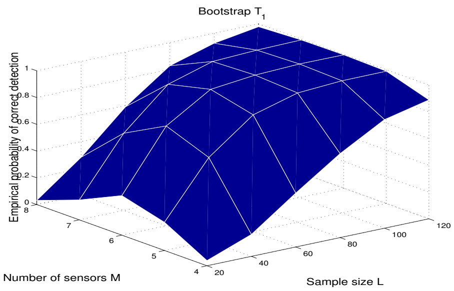
When the data or the noise is not Gaussian, as shown on Figures 2 (b) and (c), NU-MDL does not perform very well as it is derived for the particular case of stochastic data in Gaussian noise. The bootstrap detectors however do not require any assumption on the data and therefore as the SNR increases, they outperform NU-MDL. Note that performance of the bootstrap detectors is similar to the Gaussian case, indicating the robustness of the detectors.

Figure 2 (d) shows the effect of varying the number of snapshots L. As expected, performance of the three detectors improves with increasing L, as the discriminating capability of the Gerschgorin radii of the transformed sample covariance matrix, on which the detectors are based, sharpens.

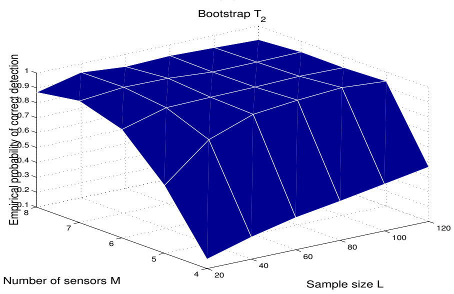
Figure 2 (e) illustrates the effect of WNPR which varies, while the other parameters are fixed. The number of snapshots is $L =$

100, and SNR=10 dB. The noise covariance matrix has the following structure

$$\mathbf{q} = [2/\text{WNPR} \;\; 1/\text{WNPR} \;\; 1 \;\; 8/\text{WNPR} \;\; 1/\text{WNPR}]^T.$$



(a)



(b)

Fig. 3. Relation between M and L and effect on the detectors' performance: **(a)** T_1, and **(b)** T_2.

The value of WNPR varies from 1 to 50. Note that when the WNPR approaches 1, the scenario is close to the special uniform noise case. This example shows the sensitivity of NU-MDL to noise nonuniformity. The bootstrap detectors perform better even in a situation where the data is Gaussian.

Figure 2 (f) illustrates the performance with respect to the angular resolution. The same parameters are employed in this example, except for SNR=10 dB, $\theta_1 = 10°$ and θ_2 is set to vary from $10°$ to $26°$. Note again that NU-MDL performs better with Gaussian data.

Figure 3 illustrates the joint effect of the number of snapshots L and the number of sensors M on the performance of the bootstrap detectors. The fixed parameters are the angles of arrival $\boldsymbol{\theta} = [25°, 35°]^T$, SNR=10 dB, and WNPR=5. The number of snapshots is set to vary from 20 to 120. The performance is evaluated for different values of M. In general, as expected, the performance improves with L increasing. Note however that for a fixed significance level α, even in asymptotic conditions, the hypothesis test stays below 100% detection rate. An important observation concerns the effect of the number of sensors M. Indeed, as the data vector $\mathbf{x}(t)$ increases in size (increasing degrees of freedom), the bootstrap requires a larger "minimal" number of snapshots to faithfully estimate the empirical distribution of the different variables. Thus, for T_1, when $L \leq 70$, the detection rate is lower with $M = 7$ than with $M = 4$. On the other hand, above a certain value ($L = 80$), the bootstrap gives better results, and the joint effect of increasing M and L improves the detection rate significantly. As explained earlier, because of its structure, T_2 shows a better performance than T_1.

5. CONCLUSION

A sequential hypothesis test for source detection has been proposed for a spatially nonuniform noise environment, when no *a priori* knowledge about the distribution of the data is available. The detector applies a transformation of the covariance matrix of the data. The proposed test statistics are based on the previously demonstrated discrimination property of the Gerschgorin radii. The distribution under the null of the test statistics is estimated using the bootstrap. Simulation results illustrate the power of the method to correctly detect the sources in various scenarios.

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