MULTICHANNEL BROADBAND FANO THEORY FOR ARBITRARY LOSSLESS ANTENNAS WITH APPLICATIONS IN DOA ESTIMATION

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ABSTRACT

In this paper we consider fundamental limitations for DOA estimation with arbitrary lossless antennas or antenna arrays inserted inside a sphere. Spherical vector modes and their associated equivalent circuits and Q factor approximations are employed as a general framework for the analysis. The classical broadband matching theory by Fano is extended to a general multiport S–parameter model of the antennas and fundamental bounds are given for the scattering parameters with respect to bandwidth and electrical size of the sphere. Finally, assuming a statistical signal model with Gaussian receiver noise, the Cramer–Rao lower bound is used to derive fundamental upper bounds for the performance of DOA estimation by a sphere.

1. INTRODUCTION

The Direction of Arrival (DOA) estimation using antenna arrays has been the topic for research in array and statistical signal processing over several decades and comprises now well developed modern techniques such as maximum likelihood and subspace methods, see e.g. [1] and the references therein. Recently, there has been an increased interest in incorporating properties of electromagnetic wave propagation with the statistical signal estimation techniques used for sensor array processing and there are several papers dealing with direction finding using electromagnetic vector sensors and diversely polarized antenna arrays, tripole arrays, etc. see e.g. [2].

The drawback of small antennas as being narrowband and lossy are well known [3, 4], and the same will of course be true for an array of antennas confined within a given volume. To analyze the estimation performance of a volume, it is essential to relate three classical theories giving fundamental limitations in the disciplines estimation theory, antenna theory and broadband matching [5]. Assuming a Gaussian signal model for the receiver noise, the Cramer–Rao bound [6] can be used as a performance measure for the estimation. The classical theory of radiating Q uses spherical vector modes and equivalent circuits to analyze the properties of a hypothetical antenna inside a sphere, c.f. [3, 4, 7].

2. SIGNAL MODEL FOR RECEIVING ANTENNAS

We consider the electromagnetic field which is propagated into free space when the transmitting antennas (all sources) are contained inside a sphere of radius $r = a$. Let $k = \omega/c$ denote the wave number, $\omega = 2\pi f$ the frequency, and c and η the speed of light and the wave impedance of free space, respectively. The transmitted electric field, $\boldsymbol{E}(\boldsymbol{r})$ can then be expanded in *outgoing spherical vector waves* $\boldsymbol{u}_{\tau ml}(k\boldsymbol{r})$ as [8]

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{\tau=1}^{2} f_{\tau ml}\boldsymbol{u}_{\tau ml}(k\boldsymbol{r}) \tag{1}$$

where $f_{\tau ml}$ are the expansion coefficients. Here $\tau = 1$ corresponds to a transversal electric (TE) wave and $\tau = 2$ corresponds to a transversal magnetic (TM) wave. The other indices are $l = 1, 2, \ldots, \infty$ and $m = -l \ldots, l$ where l denotes the *order* of that mode.

It can be shown that in the *far field* when $r \to \infty$, the electric field is given by $\boldsymbol{E}(\boldsymbol{r}) = \frac{\mathrm{e}^{-\mathrm{i}kr}}{kr}\boldsymbol{F}(\hat{\boldsymbol{r}})$ where $\boldsymbol{F}(\hat{\boldsymbol{r}})$ is the *far field amplitude* given by

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{\tau=1}^{2} \mathrm{i}^{l+2-\tau} f_{\tau ml}\boldsymbol{A}_{\tau ml}(\hat{\boldsymbol{r}}) \tag{2}$$

and $\boldsymbol{A}_{\tau ml}(\hat{\boldsymbol{r}})$ denotes the *spherical vector harmonics* [8]. Furthermore, it can also be shown that the total power P_s transmitted by the antenna can be expressed in terms of the expansion coefficients as

$$P_s = \frac{1}{2\eta k^2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{\tau=1}^{2} |f_{\tau ml}|^2. \tag{3}$$

Next, we assume that the antenna is lossless and can be modeled using a normalized multiport where a finite number of modes M is employed. As was originally described by Chu in [3], an arbitrary antenna inside a sphere of radius a can be modeled using a coupling network connecting independent equivalent circuits representing each spherical mode. The propagated power for each mode is represented by the power loss over the terminating resistance η and the wave impedance as seen by the spherical mode at radius a is equal to the input impedance of the equivalent circuit for all frequencies.

Let x_i^+ and x_i^- denote the incident and reflected voltages at the antenna waveguide connections for $i = 1, \ldots, N$ where N is the number of antenna ports. These voltages are normalized so that the power delivered to a particular antenna port is $\frac{|x_i^+|^2}{2\eta}$ and the corresponding reflected power is $\frac{|x_i^-|^2}{2\eta}$. For simplicity, we assume that the transmission line characteristic impedance is the same as the wave impedance η of free space. Each antenna port may be connected to a lossless matching network in which case y_i^+ and y_i^- denote the wave amplitudes at the antenna waveguide connections.

We let the equivalent voltage $\frac{f_\alpha}{\eta k}$ represent the propagated wave amplitude where f_α denotes the expansion coefficients for the spherical vector waves as in (1). Here, the multi–index $\alpha = (\tau, m, l)$ is chosen to simplify the notation. The multiport model is normalized to the wave impedance η and the totally transmitted power for each mode is thus equal to $\frac{1}{2\eta k^2}|f_\alpha|^2$ as in (3). The total voltage at each antenna port is denoted y_i, and the normalized equivalent voltage at the input of the TE or TM equivalent circuit is denoted by z_α/η, which is proportional to the transversal components of the electromagnetic field at radius a of the sphere [3, 8].

It is assumed that the relation between incident and reflected wave quantities can be represented by a scattering matrix as

$$\begin{pmatrix} \mathbf{z}^- \\ \mathbf{y}^- \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{z}^+ \\ \mathbf{y}^+ \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathbf{z}^+ \\ \mathbf{y}^+ \end{pmatrix} \quad (4)$$

where the matrix $\mathbf{S}$ is assumed to be lossless ($\mathbf{S}^H\mathbf{S} = \mathbf{I}$) and reciprocal ($\mathbf{S} = \mathbf{S}^T$). Furthermore, we have the following scattering parameters

$$\begin{pmatrix} \mathbf{f}^- \\ \mathbf{z}^+ \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_1' & \mathbf{T}' \\ \mathbf{T}' & \mathbf{\Gamma}_2' \end{pmatrix} \begin{pmatrix} \mathbf{f}^+ \\ \mathbf{z}^- \end{pmatrix} \quad (5)$$

and

$$\begin{pmatrix} \mathbf{y}^+ \\ \mathbf{x}^- \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_1'' & \mathbf{T}'' \\ \mathbf{T}'' & \mathbf{\Gamma}_2'' \end{pmatrix} \begin{pmatrix} \mathbf{y}^- \\ \mathbf{x}^+ \end{pmatrix} \quad (6)$$

related to the equivalent circuits and to the matching networks, respectively. All reflection and transmission matrices $\mathbf{\Gamma}$ and $\mathbf{T}$ are diagonal.

By solving (4) through (6) for $\mathbf{f}^-$ and $\mathbf{x}^-$ when $\mathbf{f}^+$ and $\mathbf{x}^+$ are given, we get the total scattering matrix

$$\begin{pmatrix} \mathbf{f}^- \\ \mathbf{x}^- \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{S}}_{11} & \bar{\mathbf{S}}_{12} \\ \bar{\mathbf{S}}_{21} & \bar{\mathbf{S}}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}^+ \\ \mathbf{x}^+ \end{pmatrix} = \bar{\mathbf{S}} \begin{pmatrix} \mathbf{f}^+ \\ \mathbf{x}^+ \end{pmatrix} \quad (7)$$

where

$$\bar{\mathbf{S}}_{11} = \mathbf{\Gamma}_1' + \mathbf{T}'\mathbf{K}^{-1}\left(\mathbf{S}_{11} + \mathbf{S}_{12}\mathbf{\Gamma}_1''\mathbf{M}^{-1}\mathbf{S}_{21}\right)\mathbf{T}' \quad (8)$$

$$\bar{\mathbf{S}}_{12} = \mathbf{T}'\mathbf{K}^{-1}\mathbf{S}_{12}\left(\mathbf{I} + \mathbf{\Gamma}_1''\mathbf{M}^{-1}\mathbf{S}_{22}\right)\mathbf{T}'' \quad (9)$$

and

$$\mathbf{K} = \mathbf{I} - \mathbf{S}_{11}\mathbf{\Gamma}_2' - \mathbf{S}_{12}\mathbf{\Gamma}_1''\mathbf{M}^{-1}\mathbf{S}_{21}\mathbf{\Gamma}_2' \quad (10)$$

$$\mathbf{M} = \mathbf{I} - \mathbf{S}_{22}\mathbf{\Gamma}_1''. \quad (11)$$

We note that the normalized multiport model described above can be interpreted as a vector two–port model which generalizes the well known result for $M = N = 1$, $\mathbf{S}_{11} = \mathbf{S}_{22} = 0$, $\mathbf{S}_{21} = \mathbf{S}_{12} = 1$, and hence

$$\bar{S}_{11} = \Gamma_1' + \frac{T'^2\Gamma_1''}{1 - \Gamma_1''\Gamma_2'} \quad (12)$$

$$\bar{S}_{12} = \frac{T'T''}{1 - \Gamma_1''\Gamma_2'} \quad (13)$$

cf. e.g. [5]. Here $|\bar{S}_{11}|^2 + |\bar{S}_{21}|^2 = 1$ and $\bar{S}_{21} = \bar{S}_{12}$.

Next, we derive the multiport scattering model for receiving antennas by considering the reciprocity theorem. On transmission the transmitted wave field $\mathbf{f}^-$ is given by $\mathbf{f}^- = \bar{\mathbf{S}}_{12}\mathbf{x}^+$. Thus, if we consider the transmitted wave field f_α due to one single input terminal with the incident voltage wave x_i^+, we get the output $f_\alpha = k\left[\bar{\mathbf{S}}_{12}\right]_{\alpha,i} x_i^+$. Now, from the antenna reciprocity theorem [9] we have

$$x_i^- x_i^+ = -\mathrm{i}\frac{\lambda^2}{2\pi}\boldsymbol{F}(\hat{\boldsymbol{k}}_0)\cdot\boldsymbol{E}_0 \quad (14)$$

where $\boldsymbol{E}_0$ is the complex vector amplitude of an incoming plane wave from direction $\hat{\boldsymbol{k}}_0$ and x_i^- the corresponding received signal. Further, $\boldsymbol{F}(\hat{\boldsymbol{r}})$ is the far field amplitude corresponding to the transmitted signal x_i^+. Hence, by using (2) the received signal is obtained from the reciprocity theorem (14) as

$$\mathbf{x}^- = \frac{2\pi}{k}\bar{\mathbf{S}}_{21}\mathbf{A}\mathbf{E} \quad (15)$$

where $\mathbf{A}$ is an $M \times 2$ matrix where each row corresponds to the spherical components of the spherical vector harmonics $\mathrm{i}^{l+1-\tau}\boldsymbol{A}_\alpha(\hat{\boldsymbol{k}}_0)$, and $\mathbf{E}$ is an 2×1 vector containing the corresponding signal components of the electric field $\boldsymbol{E}_0$.

Now, from (15), a complex baseband model for the received signal is given by

$$\mathbf{x}(t) = \frac{2\pi}{k}\bar{\mathbf{S}}_{21}\mathbf{A}\mathbf{E} + \mathbf{n}(t) \quad (16)$$

where $\mathbf{n}(t)$ is white complex Gaussian noise with covariance matrix $\sigma_n^2\mathbf{I}$. We consider a situation where the received electric field is monochromatic and completely polarized. We assume a narrowband signal model where k corresponds to the carrier frequency ω_0 and the *fractional bandwidth* $B = \frac{\Delta\omega}{\omega_0}$ is reasonable low. Here $\Delta\omega$ denotes the absolute bandwidth and $\sigma_n^2 = N_0\omega_0 B$ where N_0 is the spectral density of the Gaussian process.

We are interested in the estimation accuracy of the spherical DOA parameters θ and ϕ which we write as a vector parameter $\boldsymbol{\xi} = [\theta\ \phi]^T$. The Fisher Information matrix [6] becomes

$$[\mathbf{I}(\boldsymbol{\xi})]_{ij} = \frac{8\pi^2}{\sigma_n^2 k^2} \operatorname{Re}\left\{\mathbf{p}_i^H \bar{\mathbf{S}}_{21}^H \bar{\mathbf{S}}_{21} \mathbf{p}_j\right\} \tag{17}$$

where

$$\mathbf{p}_i = \frac{\partial}{\partial \xi_i} \{\mathbf{AE}\}. \tag{18}$$

Now, the variance of each parameter is bounded as

$$\operatorname{var}\left\{\hat{\xi}_i\right\} \geq \left[\mathbf{I}^{-1}(\boldsymbol{\xi})\right]_{ii} \geq \frac{1}{[\mathbf{I}(\boldsymbol{\xi})]_{ii}} \tag{19}$$

where $[\mathbf{I}(\boldsymbol{\xi})]_{ii}$ is given by

$$[\mathbf{I}(\boldsymbol{\xi})]_{ii} = \frac{8\pi^2}{\sigma_n^2 k^2} \mathbf{p}_i^H \bar{\mathbf{S}}_{21}^H \bar{\mathbf{S}}_{21} \mathbf{p}_i \tag{20}$$

which is real and nonnegative. Since $\bar{\mathbf{S}}$ is lossless we have $\bar{\mathbf{S}}_{21}^H \bar{\mathbf{S}}_{21} + \bar{\mathbf{S}}_{11}^H \bar{\mathbf{S}}_{11} = \mathbf{I}$ and the eigenvalues of $\bar{\mathbf{S}}_{21}^H \bar{\mathbf{S}}_{21}$ are in the interval $[0, 1]$. It is therefore concluded that

$$[\mathbf{I}(\boldsymbol{\xi})]_{ii} \leq \frac{8\pi^2}{\sigma_n^2 k^2} \mathbf{p}_i^H \mathbf{p}_i. \tag{21}$$

Hence, $\operatorname{var}\left\{\hat{\xi}_i\right\}$ in (19) is bounded below by

$$\operatorname{var}\left\{\hat{\xi}_i\right\} \geq \frac{1}{[\mathbf{I}(\boldsymbol{\xi})]_{ii}} \geq \frac{k^2 N_0 \omega_0}{8\pi^2} F_{\mathrm{a}} \tag{22}$$

where we have defined the *accuracy factor* for general antennas

$$F_{\mathrm{a}} = \frac{B}{\mathbf{p}_i^H \mathbf{p}_i}. \tag{23}$$

We note also that for the idealized mode–coupled antenna where $\bar{\mathbf{S}}_{11}$ is diagonal, the CRLB expression (20) can be calculated when the reflection coefficients $|\Gamma_j|^2$ are known

$$[\mathbf{I}(\boldsymbol{\xi})]_{ii} = \frac{8\pi^2}{\sigma_n^2 k^2} \mathbf{p}_i^H \left(\mathbf{I} - \bar{\mathbf{S}}_{11}^H \bar{\mathbf{S}}_{11}\right) \mathbf{p}_i. \tag{24}$$

The final bound for $\operatorname{var}\left\{\hat{\xi}_i\right\}$ in (19) becomes

$$\operatorname{var}\left\{\hat{\xi}_i\right\} \geq \frac{1}{[\mathbf{I}(\boldsymbol{\xi})]_{ii}} = \frac{k^2 N_0 \omega_0}{8\pi^2} F_{\mathrm{a}}^{\mathrm{CRLB}} \tag{25}$$

where the Cramer–Rao lower bound accuracy factor is

$$F_{\mathrm{a}}^{\mathrm{CRLB}} = \frac{B}{\mathbf{p}_i^H \operatorname{diag}\left[1 - |\Gamma_j|^2\right] \mathbf{p}_i}. \tag{26}$$

3. BROADBAND FANO–THEORY FOR THE MULTIPORT MODEL

In this section we show that some of the important theoretical limitations for two–port broadband matching of arbitrary impedances as given by Fano in [5], can be generalized to the multiport model described in the previous section.

Consider the scattering matrix $\bar{\mathbf{S}}_{11}$ given in (8) and assume that the diagonal elements T'_j of the transmission coefficient $\mathbf{T}'$ has a *common* zero at $s = 0$ with multiplicity n. Denote the diagonal elements of $\bar{\mathbf{S}}_{11}$ by Γ_j and the elements of $\boldsymbol{\Gamma}'_1$ by Γ'_j. The Taylor series expansion of the logarithm of the diagonal elements Γ_j about $s = 0$ can then be written

$$\log \frac{1}{\Gamma_j} = A_1 s + \cdots + A_{2k+1} s^{2k+1} + \cdots + A_{2n-1} s^{2n-1} + \cdots \tag{27}$$

where even order coefficients up to and including $2n-2$ are zero, and the odd coefficients A_{2k+1} are independent of the matching network ($\boldsymbol{\Gamma}''_1$, $\boldsymbol{\Gamma}''_2$, $\mathbf{T}''$) for $k = 0, 1, \ldots, n-1$. These facts can be established by following the derivation in [5] using (8) and noting that $s = 0$ is a common zero of $\mathbf{T}'$ of multiplicity n. By employing the calculus of residues, integral relations are then obtained as in [5] that relate the reflection coefficient Γ_j (over bandwidth B) to the zeros s'_{oi} and poles s'_{pi} of Γ'_j.

In theory, the equivalent circuits can be used to derive a Fano limit for any TE or TM mode. However, instead of using the analytic expressions of the impedance it is common to use the Q factor to get an estimate of the bandwidth [3, 4, 7]. At and around the resonance frequency, $\omega_0 = 2\pi f_0$, the antenna model is given by a resonance circuit. The resonance circuit is either a series RCL circuit with capacitance $\frac{1}{Q\omega_0}$ and inductance $\frac{Q}{\omega_0}$, or a parallel circuit with these values switched. The transmission coefficients $T'_j(s)$ for the Q factor resonance circuits have a single zero at $s = 0$ and a single zero at $s = \infty$ (common for all modes). The reflection coefficients $\Gamma'_j(s)$ have zeros at $s'_{oi} = \pm \mathrm{i}\omega_0$ and poles $s'_{pi} = \frac{\omega_0}{Q}(-1 \pm \mathrm{i}\sqrt{Q^2 - 1})$.

By assuming a constant reflection coefficient $|\Gamma_j|$ over the bandwidth $[\omega_0 - \omega_0 \frac{B}{2}, \omega_0 + \omega_0 \frac{B}{2}]$ and introducing the constant $K = \frac{2}{\pi} \log \frac{1}{|\Gamma_j|}$, the two integrals in [5] for $k = 0$ become

$$\frac{KB}{1 - B^2/4} = \frac{2}{Q} - 2\sum \frac{\omega_0}{s_{ri}} \tag{28}$$

$$KB = \frac{2}{Q} - 2\sum \frac{s_{ri}}{\omega_0} \tag{29}$$

where s_{ri} are the zeros of Γ_j in the right half–plane.

We can see that these equations can be satisfied by one complex conjugated pair s_{ri} and s^*_{ri} as follows. Let $\frac{s_{ri}}{\omega_0} = x + \mathrm{i}y$, then $\operatorname{Re}\left\{\frac{s_{ri}}{\omega_0}\right\} = x$ and $\operatorname{Re}\left\{\frac{\omega_0}{s_{ri}}\right\} = \frac{x}{x^2+y^2}$. Since $KB < KB/(1 - B^2/4)$, the equations can be satisfied by letting $y \to \infty$ and then chosing a suitable $x > 0$. Hence, the relation (28) gives an inequality which is a greatest lower bound for $|\Gamma_j|$

$$|\Gamma_j| \geq \mathrm{e}^{-\frac{\pi}{Q}\frac{1-B^2/4}{B}}. \tag{30}$$

4. NUMERICAL EXAMPLES

In Fig. 1 is shown the optimum reflection coefficient $|\Gamma_j|$ for the first 3 mode orders $n = 1, 2, 3$ as a function of the electrical size ka when $B = 0.01$. For a given bandwidth B, all modes will ultimately be useless (useful), i.e. $|\Gamma|$ will approach unity (zero) as the electrical size ka decreases (increases). For a given electrical size ka, there is always a certain limited number of modes that are useful with $|\Gamma|$ significantly less than unity.

In Fig. 2 is shown the accuracy factor F_a and $F_\mathrm{a}^\mathrm{CRLB}$ for DOA estimation given in (23) and (26) (with $\xi = \theta$ and $\theta = 0$) for the first 3 mode orders $n = 1, 2, 3$, as a function of the electrical size ka when $B = 0.01$. As the electrical size ka decreases, the accuracy of DOA estimation is determined by a decreasing number of modes. In this example with $B = 0.01$, it is sufficient to consider 3 mode orders for $ka = 1$, 2 mode orders for $ka = 0.5$ and 1 mode order for $ka = 0.1$.

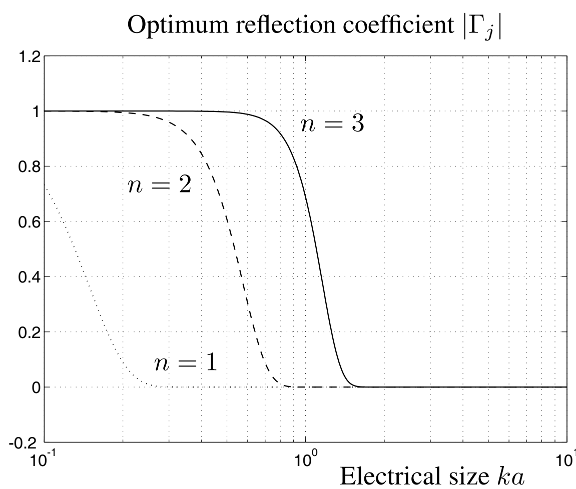

Fig. 1. Optimum reflection coefficient $|\Gamma_j|$ for the first 3 mode orders $n = 1, 2, 3$ as a function of electrical size ka. Fractional bandwidth is $B = 0.01$. Solid line: $n = 3$. Dashed line: $n = 2$. Dotted line: $n = 1$.

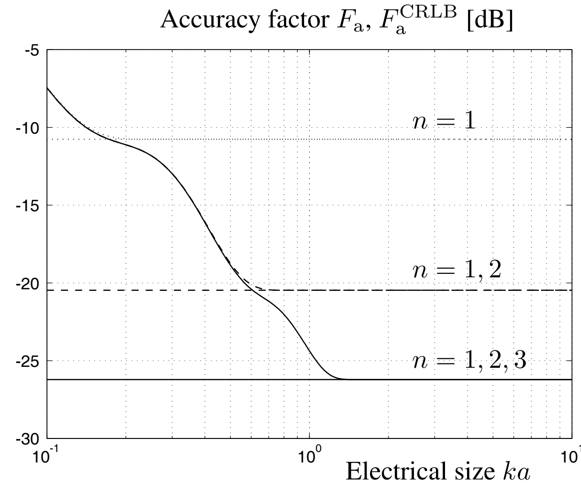

Fig. 2. Accuracy factor F_a and $F_\mathrm{a}^\mathrm{CRLB}$ for the first 3 mode orders $n = 1, 2, 3$ as a function of electrical size ka. Fractional bandwidth is $B = 0.01$. Solid line: Three mode orders included $n = 1, 2, 3$. Dashed line: Two mode orders included $n = 1, 2$. Dotted line: One mode order included $n = 1$.

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