ON THE AMBIGUITY OF COMET-EXIP ALGORITM FOR ESTIMATING A SCATTERED SOURCE

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ABSTRACT

The EXtended Invariance Principle (EXIP) has been applied to the structured covariance estimation of a zero mean Gaussian vector, the resulting method was named COMET (COvariance Matching Estimation Techniques). This technique has been recently used for estimating separately and efficiently the direction of arrival (DOA) and angular spread of a scattered source. Unfortunately this new technique presents an ambiguity that limits its utilization in practice. We show in this paper the existence and the origin of this ambiguity and we propose a solution to eliminate this problem without introducing bias. Our approach consists first to add a constraint to the original cost function, and then to replace the constrained problem by an unconstrained problem by using the penalty function method.

1. INTRODUCTION

Traditional direction-finding techniques have generally been developed for far-field point sources which travel along a single path to the antenna array. However, in applications such as mobile communications and sonar where the effect of angular spread can not be ignored, due to multipath phenomena, a distributed source model will be more appropriate [3].

Many estimators for spatially distributed sources have recently been proposed. In [5] and [4] respectively, the authors have proposed the DSPE and DISPAR subspace-based methods that are based on the eigen-decomposition of the sample covariance matrix. Simulation results have shown a performances degradation of these two techniques especially in the presence of multiple scattered sources. For the purpose to improve the performances of these techniques, the authors of [1] present a generalization of the WSF method (Weighted Subspace Fitting) in the case of full-rank data model. This method, called WPSF [1], is based on the eigen-decomposition of the covariance matrix into the signal and noise subspaces of variable size. The major problem of this technique is the determination of the number of eigenvectors that span the signal and noise subspaces. Moreover, all these methodes require an important computational time.

In this paper, we focus our attention on the estimation of DOA and angular spread of a spatially distributed source by using a statistically and computationally efficient algorithm (decoupled COMET-EXIP estimator also called EXIP-based estimator) [2]. This method estimates separately the DOA and the angular spread of scattered source. This enables to replace a two-dimensional minimization problem by two successive and simple one-dimensional minimization problem. Another advantage of this technique, is that the DOA estimation does not require any knowledge of the distribution of scatterers around the nominal direction. Estimation of spread angle, however, requires this knowledge.

Unfortunately, the pseudo-spectrum of this technique presents an ambiguity. This prevents its utilization in practice. The aim of this paper is first, to find the origin of this ambiguity problem and then to propose an approach for solving this ambiguity. Our approach consists first to add a constraint to the original COMET-EXIP cost function, and then to replace the constrained problem by an unconstrained problem by using the penalty function method.

2. PROBLEM FORMULATION

In this paper we consider a uniform linear array (ULA) of m sensors. The distance between two adjacent sensors is denoted by d. Suppose that an electromagnetic scattered wave is impinging on the array from angular direction θ_o. The distributed model proposed in [1] is adopted in this paper. For this model, the received signal is written as

$$\mathbf{x}(t) = s(t) \sum_{n=1}^{N} \alpha_n(t)\mathbf{a}(\theta_o + \tilde{\theta}_n(t)) + \mathbf{n}(t) \qquad (1)$$

where $s(t)$ is the emitted signal and $\mathbf{n}(t)$ denotes the noise vector. N, $\alpha_n(t)$, θ_o and $(\theta_o + \tilde{\theta}_n)$, are the number of reflectors surrounding the source, complex gain of the n^{th} scattered signal, nominal DOA of the source and the DOA of the n^{th} scattered signal, respectively. The gains are assumed to be independent from ray to ray. The steering vector for a point source at DOA θ is denoted by $\mathbf{a}(\theta) = [1, e^{j\frac{2\pi d}{\lambda}sin\theta}, ..., e^{j\frac{2\pi d}{\lambda}(m-1)sin\theta}]^T$, where, $(.)^T$ denotes the transpose operator and λ is the wavelength of the impinging signal. As in [1], the scattering environment changes rapidly compared with the mean DOA and spread parameters. In other words, the random complex gains $\alpha_n(t)$ are assumed to be temporally white, zero-mean :

$$\begin{aligned} E\{\alpha_n(t)\} &= 0 \\ E\{\alpha_n(t)\alpha^*_{n'}(t')\} &= \frac{\sigma^2_\alpha}{N}\delta(n,n')\delta(t,t') \end{aligned} \quad (2)$$

where σ^2_α is the path power factor. The noise $\mathbf{n}(t)$ is considered as a circular complex Gaussian random variable, zero-mean and spatio-temporally white :

$$E\{\mathbf{n}(t)\mathbf{n}^H(t')\} = \sigma^2_n \mathbf{I}\delta(t,t') \quad (3)$$

$E\{.\}$ denotes the expectation and $(.)^H$ denotes the complex conjugate transpose operator and $(.)^*$ is the conjugate.

Assuming that the noise and signal are uncorrelated, the data model (1) allows us to write the covariance matrix of the array measurements as :

$$\mathbf{R} = E[\mathbf{x}(t)\mathbf{x}^H(t)] = \mathbf{\Phi}(\theta_o)\mathbf{B}\mathbf{\Phi}^H(\theta_o) \quad (4)$$

where $\mathbf{\Phi}(\theta) = diag\{\mathbf{a}(\theta)\}$ and $\mathbf{B}$ is given by

$$\mathbf{B} = \sigma^2_{s\alpha}\int p(\tilde{\theta})\mathbf{a}(\tilde{\theta})\mathbf{a}^H(\tilde{\theta})\,d\tilde{\theta} + \sigma^2_n\mathbf{I} \quad (5)$$

with $\sigma^2_{s\alpha}{=}E\{\mid s(t)\mid^2\}E\{\mid \alpha_n(t)\mid^2\}$ is the signal source power including the path gain factor and $p(\tilde{\theta})$ is the angular distribution density of reflectors. It is easy to show that $\mathbf{B}$ is a real-valued symmetric Toeplitz matrix. It is uniquely determined by the elements of its first column vector denoted by $\boldsymbol{\beta} = [\beta_o, \beta_1, \ldots, \beta_{m-1}]^T$. Its elements depend, as the formula (7) indicates, on the type of distribution of reflectors around the source. For $0 \leq q \leq m-1$, we have

$$\beta_q = \beta_q(\theta_o, \sigma_o) = \sigma^2_{s\alpha}\epsilon_q + \sigma^2_n\delta(q,0) \quad (6)$$

with

$$\epsilon_q \simeq \begin{cases} e^{-\frac{1}{2}(q\frac{2\pi d}{\lambda}\sigma_o\cos\theta_o)^2} & \text{Gaussian dist.} \\ sinc(\sqrt{3}q\frac{2\pi d}{\lambda}\sigma_o\cos\theta_o) & \text{uniform dist.} \end{cases} \quad (7)$$

The parameter σ_o represents the standard deviation of $\tilde{\theta}$.

3. DECOUPLED COMET-EXIP ESTIMATOR

We present briefly in this section the methods proposed in [2], in which the considered problem is to estimate the parameters vector $\boldsymbol{\eta} = [\theta \quad \sigma \quad \sigma^2_{s\alpha} \quad \sigma^2_n]^T$. For the sake of convenience, this parameter vector can be reparametrized as $\boldsymbol{\eta} = [\theta \quad \boldsymbol{\beta}^T]^T$. The COMET estimate is obtained by minimizing the following cost function

$$C(\boldsymbol{\eta}) = \left\| \mathbf{W}^{H/2}\big[\hat{\mathbf{R}} - \mathbf{R}(\boldsymbol{\eta})\big]\mathbf{W}^{1/2}\right\|^2 \quad (8)$$

where $\mathbf{W}$ is a positive definite weighting matrix and $\hat{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^T \mathbf{x}(t)\mathbf{x}^H(t)$ is the sample covariance matrix. By using the fact that $\mathbf{Vec}(\mathbf{B}) = \mathbf{J}\boldsymbol{\beta}$, where $\mathbf{J}$ is a $m^2 \times m$ logical matrix such as $\mathbf{J}((n-1)m+l, k) = 1$ for $\mid l-n \mid = k-1$ and for $1 \leq n, l, k \leq m$, and $\mathbf{Vec}(.)$ is the vector obtained by stacking the columns of the argument on top of each other, the COMET cost function (8), can be transformed into the following cost function

$$C(\theta,\boldsymbol{\beta}){=}\mathbf{Vec}^H(\hat{\mathbf{R}})\breve{\mathbf{W}}\mathbf{Vec}(\hat{\mathbf{R}}) - 2\boldsymbol{\beta}^T\boldsymbol{y} + \boldsymbol{\beta}^T\boldsymbol{Y}\boldsymbol{\beta} \quad (9)$$

where $\breve{\mathbf{W}} = \mathbf{W}^T \otimes \mathbf{W}$ with $\otimes$ denotes the Kronecker matrix product, $\boldsymbol{y} = \mathbf{J}^T\Psi^H(\theta)\breve{\mathbf{W}}\mathbf{Vec}(\hat{\mathbf{R}})$ and the matrix $\boldsymbol{Y} = \mathbf{J}^T\Psi^H(\theta)\breve{\mathbf{W}}\Psi(\theta)\mathbf{J}$ with $\Psi(\theta) = \mathbf{\Phi}^H(\theta) \otimes \mathbf{\Phi}(\theta)$

According to [2], the optimal vector $\hat{\boldsymbol{\beta}}$ that minimizes (9) and the optimal cost function $\tilde{C}(\theta)$ obtained by replacing $\hat{\boldsymbol{\beta}}$ into (9) are given by

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \boldsymbol{Y}^{-1}\boldsymbol{y} & (10) \\ \tilde{C}(\theta) &= K_{\mathbf{W}} - \boldsymbol{y}^T\boldsymbol{Y}^{-1}\boldsymbol{y} & (11) \end{aligned}$$

where $K_{\mathbf{W}} = \mathbf{Vec}^H(\hat{\mathbf{R}})\breve{\mathbf{W}}\mathbf{Vec}(\hat{\mathbf{R}})$. Note that, the first term of (11), $K_{\mathbf{W}}$, is constant. Thus, the solution is also given by the following maximization problem [2],

$$\hat{\theta} = arg\,\max_\theta\,\boldsymbol{y}^T\boldsymbol{Y}^{-1}\boldsymbol{y} \quad (12)$$

In order to reduce the computational load associated with the maximization of the above function, the authors propose to use the unweighted cost function (replacing the matrix $\mathbf{W}$ by an identity matrix in (12)) to find an initial estimate of the DOA. Then, the solution is refined by using an iterative search method (see [2]).

4. AMBIGUITY PROBLEM

In order to see the ambiguity of the COMET-EXIP method, it is sufficient to show that there exists always a direction, different to the effective direction of the source, for which the value of the unweighted cost function is the same as that for the true direction. The same steps can be made to show the ambiguity of the weighted cost function, but in this case the calculations will be more complicated.

For the unweighted cost function ($\mathbf{W} = \mathbf{I}$), the vector $\boldsymbol{y}$ becomes $\boldsymbol{y}_I = \mathbf{J}^T\Psi^H\mathbf{Vec}(\hat{\mathbf{R}})$ of which the first element is $[\boldsymbol{y}_I]_1 = \sum_{k=1}^m \hat{\mathbf{R}}(k,k)$ and the others elements ($1 \leq k \leq m-1$) are given by $[\boldsymbol{y}_I]_{k+1} = 2Re\big[\chi_k e^{-j\frac{2\pi k d}{\lambda}\sin\theta}\big]$ where, $\chi_k = \sum_{s=1}^{m-k}\hat{\mathbf{R}}(s+k,s)$. Moreover, we have

$\boldsymbol{Y}_I^{-1} = (\mathbf{J}^T\mathbf{J})^{-1} = diag[1/m,\ 1/2(m-1),\ ...,\ 1/2]$. Then, the unweighted cost function can be expressed as

$$\tilde{C}_I(\theta) = K_{\mathbf{I}} - \boldsymbol{y}_I^T \boldsymbol{Y}_I^{-1} \boldsymbol{y}_I \tag{13}$$

$$= K_{\mathbf{I}} - \sum_{k=0}^{m-1} \frac{|\chi_k|^2}{m-k} - Re\left(\sum_{k=1}^{m-1} \frac{\chi_k^2}{m-k} e^{-j\frac{4\pi k d}{\lambda}\sin\theta} \right) \tag{14}$$

where $K_{\mathbf{I}} = \mathbf{Vec}^H(\hat{\mathbf{R}})\mathbf{Vec}(\hat{\mathbf{R}})$. By using the property

$$\lim_{T\to\infty} \chi_k = \sigma_{s\alpha}^2 (m-k) \beta_k(\theta_o, \sigma_o) e^{j\frac{2k\pi d}{\lambda}\sin\theta_o} \tag{15}$$

where $\beta_k(\theta_o, \sigma_o)$ is given by (6) and (θ_o, σ_o) are the true DOA and spread of the source. It is easy to show from (15) and (14) that, for $d = \lambda/2$, the value of $\tilde{C}_I(\theta)$ is the same one for the true DOA θ_o and for $\theta = sin^{-1}(\sin\theta_o + p)$ with $p \in \mathbb{Z}$. This equation has real solutions for : $\sin\theta_o + p \in [-1, 1]$. Thus, the possible values of p are -1 , 0 and 1 :

- The value $p = 1$ corresponds to $\theta_o \in [-90°, 0°[$
- The value $p = -1$ corresponds to $\theta_o \in\ ]0°, 90°]$
- The value $p = 0$ corresponds to the true direction $\theta = \theta_o$

As a conclusion, for a fixed value of θ_o, there is always an ambiguity direction defined by the equation :

$$\theta_{amb} = \sin^{-1}\left[\sin(\theta_o) - sgn(\theta_o)\right] \tag{16}$$

where $sgn(\theta_o)$ is the Signum function defined for nonzero direction θ_o by $sgn(\theta_o) = \theta_o / |\theta_o|$.

5. PROBLEM RESOLUTION

In order to find the origin of this ambiguity problem, we compare the elements of the vector $\hat{\boldsymbol{\beta}}_I = \boldsymbol{Y}^{-1}\boldsymbol{y}$ for the ambiguity direction θ_{amb}(16) with these for the true direction θ_o in the case where $d = \lambda/2$. The elements of this vector are given by

$$\left[\hat{\boldsymbol{\beta}}_I(\theta)\right]_{k+1} = \frac{Re\left(\chi_k\, e^{-jk\pi\sin\theta}\right)}{m-k} \tag{17}$$

where $k = \{0,\ 1,\ ...,\ m-1\}$. Thus,

$$\left[\hat{\boldsymbol{\beta}}_I(\theta_{amb})\right]_{k+1} = (-1)^k \left[\hat{\boldsymbol{\beta}}_I(\theta_o)\right]_{k+1} \tag{18}$$

As we can see, the elements of the vector $\hat{\boldsymbol{\beta}}_I$ for the true direction are equal in absolute value to these for the ambiguity direction. On the other hand, the odd values of k change the sign by passing the true direction to the ambiguity direction. A graphic illustration of the elements of vector $\hat{\boldsymbol{\beta}}$ (10) for the Gaussian distribution of reflectors is given in

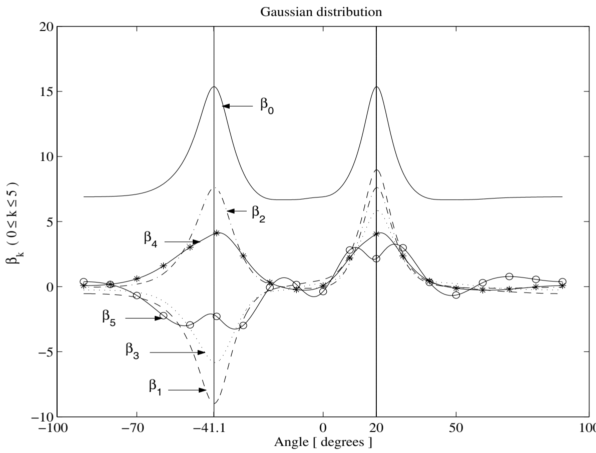

Fig. 1. Ambiguity effect on the different elements of the vector $\hat{\boldsymbol{\beta}}$ given by (10), $\theta_o \simeq 20°$, $\theta_{amb} \simeq -41.1°$, $\sigma = 5°$, $m = 6$ and $SNR = 10\ dB$ (Gaussian distribution).

figure 1, the same behavior can be observed for the uniform distribution.

From this result, it is easy to see that it is necessary to add a constraint to the minimization of the cost function (9) in order to eliminate this ambiguity. According to equation (6), it is clear that for a Gaussian distribution of reflectors around the source, all elements of the vector $\hat{\boldsymbol{\beta}}$ have to be positive. This condition remains true for small values of the angular spread of uniform distribution. Thus, the new minimization problem can be expressed as

$$\min_{\boldsymbol{\beta}}\ C(\theta, \boldsymbol{\beta}) \ \text{ subject to } \ \boldsymbol{\beta} \geq 0 \tag{19}$$

where $C(\theta, \boldsymbol{\beta})$ is the COMET cost function (9).

This constrained minimization problem, can be transformed into the following unconstrained minimization problem

$$\min_{\boldsymbol{\beta}}\ C(\theta, \boldsymbol{\beta}) + \zeta\, \mathbf{1}_m^T \boldsymbol{g}_2(\boldsymbol{\beta}) \tag{20}$$

where $C(\theta, \boldsymbol{\beta})$ is called objective function, the non-negative value ζ is the penalty parameter, $\left[\boldsymbol{g}_2\right]_{k+1} = \left[max(-\beta_k, 0)\right]^2$ for $0 \leq k \leq m-1$ is the k^{th} penalty function and $\mathbf{1}_m = [1, 1, ..., 1]^T$. This function must be continuous and positive. For the sake of convenience, we introduce the following auxiliary function :

$$\psi(\theta, \boldsymbol{\beta}, \zeta) = C(\theta, \boldsymbol{\beta}) + \zeta\, \mathbf{1}_m^T \boldsymbol{g}_2(\boldsymbol{\beta})$$
$$= K_{\mathbf{W}} - 2\boldsymbol{\beta}^T \boldsymbol{y} + \boldsymbol{\beta}^T \boldsymbol{Y} \boldsymbol{\beta} + \zeta\, \mathbf{1}_m^T \boldsymbol{g}_2(\boldsymbol{\beta}) \tag{21}$$

Differentiating this function with respect to the vector $\boldsymbol{\beta}$,

$$\frac{\partial\psi(\theta, \boldsymbol{\beta}, \zeta)}{\partial\boldsymbol{\beta}} = -2\boldsymbol{y} + 2\boldsymbol{Y}\boldsymbol{\beta} - 2\,\zeta\, \boldsymbol{g}_1(\boldsymbol{\beta}) \tag{22}$$

where $\left[\boldsymbol{g}_1\right]_{k+1} = max(-\beta_k, 0)$. According to the value of $\hat{\boldsymbol{\beta}} = \boldsymbol{Y}^{-1}\boldsymbol{y}$ given by (10), if there exists a k such as $\hat{\beta}_k < 0$. The vector that cancels the equation (22) is given by :

$$\hat{\boldsymbol{\beta}} = [\,\boldsymbol{Y}(\theta) + \zeta \mathbf{D}_k\,]^{-1}\,\boldsymbol{y} \tag{23}$$

with $\mathbf{D}_k = diag[0, \ldots, 0, 1, 0, \ldots, 0]$, the one is in the k^{th} position. By using the eigen-decomposition of the real-valued matrix $\boldsymbol{Y}(\theta)$, we obtain

$$[\,\boldsymbol{Y}(\theta) + \zeta \mathbf{D}_k\,]^{-1} = \mathbf{U}\,[\,\boldsymbol{\Sigma} + \zeta \mathbf{D}_k\,]^{-1}\,\mathbf{U}^T \tag{24}$$

where $\boldsymbol{\Sigma} = diag[\mu_0, \mu_1, \ldots, \mu_{m-1}]$ is the eigenvalues matrix of $\boldsymbol{Y}(\theta)$ and $\mathbf{U} = [\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{m-1}]$ the corresponding eigenvectors matrix. Outside the admissible region, where the inequality constraint is violated, the penalties become infinite. Thus, the optimal solution of the penalty problem can be made arbitrarily close to the solution of the original problem by choosing ζ sufficiently large. Therefore, the optimal solution is given by :

$$\hat{\boldsymbol{\beta}}_{opt} = \mathbf{U}\mathbf{M}_k\mathbf{U}^T\boldsymbol{y} \tag{25}$$

$$\text{with } \mathbf{M}_k = \lim_{\zeta\to\infty}\,[\,\boldsymbol{\Sigma} + \zeta \mathbf{D}_k\,]^{-1}$$

$$= diag\left[\mu_0^{-1}, \ldots, \mu_{k-1}^{-1}, 0, \mu_{k+1}^{-1}, \ldots, \mu_{m-1}^{-1}\right] \tag{26}$$

The same calculation can be made if multiple elements of the vector $\hat{\boldsymbol{\beta}}$ are negative. In this case, the optimal solution can expressed as :

$$\hat{\boldsymbol{\beta}}_{opt} = \mathbf{U}\mathbf{M}\mathbf{U}^T\boldsymbol{y} \tag{27}$$

with $\mathbf{M} = diag\left[\frac{H(\hat{\beta}_0)}{\mu_0}, \frac{H(\hat{\beta}_1)}{\mu_1}, \ldots, \frac{H(\hat{\beta}_{m-1})}{\mu_{m-1}}\right]$, and $H(.)$ is defined as $H(\beta) = 1$ if $\beta \geq 0$ and $H(\beta) = 0$ else. Inserting (27) into (9), the DOA of source is given by minimizing the following modified cost function :

$$C_{corr}(\theta) = K_{\mathbf{W}} - \boldsymbol{y}^T\mathbf{U}\mathbf{M}\mathbf{U}^T\boldsymbol{y} \tag{28}$$

The comparison between the pseudo-spectrum of this last cost function (called Corrected COMET-EXIP) with the one given by (11) is illustrated in the figure 2. This figure shows the efficiency of our method to eliminate the ambiguity problem. We don't present the perfomance of the proposed method, because this method has the same estimation performance as the original COMET-EXIP method. The advantage of our method is that, it is not ambiguous.

6. CONCLUSION

In this paper, we have presented the ambiguity problem of the decoupled COMET-EXIP algorithm. To solve this problem, we have shown that it is necessary to add an inequality constraint in the original cost functions of COMET-EXIP estimator. In order to solve this minimization problem, we have used the penalty function method to transform a constrained problem into an unconstrained problem. The inequality constraint is placed into the objective function via a penalty parameter in such a way that it penalizes any violation of the constraint. The final decoupled cost function can be seen as a generalization of the original decoupled cost function. Using this function, it is possible to localize the source without ambiguity and without introducing bias.

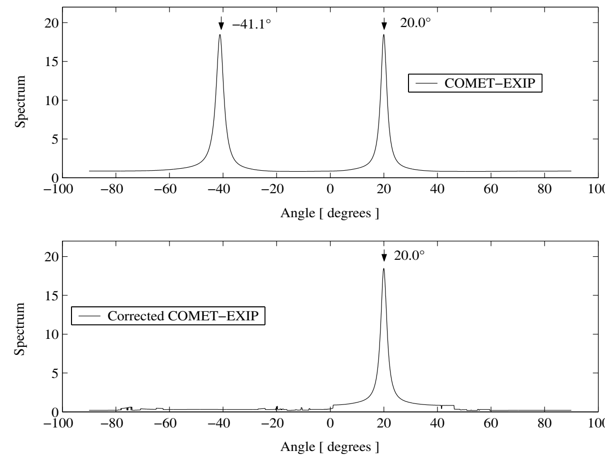

Fig. 2. COMET-EXIP (12) and Corrected COMET-EXIP (28) spectra, $\theta_o \simeq 20°$, $\sigma = 5°$, $m = 6$ and $SNR = 10\,dB$.

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